Quaternions, the 3-sphere & it's Quotients

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Among all 3-dimensional manifolds, the 3-sphere $\mathbb{S}^3 = \{v \in \mathbb{R}^4 : \|v\| = 1\}$ is perhaps the most emblematic example. Although conceptually simple, the 3-sphere – just like most manifolds – cannot be embedded in $\mathbb{R}^3$. In other words, the 3-sphere is not *drawable*. This is a problem for us, mortals condemned to a 3-dimensional (local) existence who seek to understand $\mathbb{S}^3$'s geometric structure through our visual intuition.

That said, the mathematical community at large has been developing different *ways* to visualize the 3-spheres since the mid 19th centuries. Among these, the study of it's Lie group structure stands out. Although not that visual, this approach is very practical and it allows us extract a lot of information about the differentiable structure of $\mathbb{S}^3$ and that of many of it's quotients. But wait… $\mathbb{S}^3$ is a group? Why?

The central ingredient of the following discussion is that of the quaternion numbers $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$ – where $i^2 = j^2 = k^2 = ijk = -1$. The quaternions are hypercomplex numbers – an "extension" of the complex plane – of course, but what's reinvent to us is that they form a 4-dimensional normed division algebra over $\mathbb{R}$. In particular, given $q, p \in \mathbb{H}$ with $|q| = |p| = 1$, $|qp| = |q| \cdot |p| = 1$. In other words, the set of unitary quaternion numbers is closed under multiplication.

Since $\mathbb{H}$ is a division algebra, the set of unitary quaternions is a group. Moreover, it's easy to show that quaternion multiplication is polynomial – i.e. it is a polynomial function in each coordinate. This implies that the unitary quaternions are a Lie group. Now by identifying $\mathbb{R}^4$ with $\mathbb{H}$ we arrive at

$$\mathbb{S}^3 = \{q \in \mathbb{H} : |q| = 1\}$$

and we finally conclude that $\mathbb{S}^3$ is a Lie group under the quaternion product.

Thoughtful readers may have already realized that this is precisely the way we endow the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with it's group structure. Indeed, the same exact argument works for dimensions 0 and 1 too – simply replace $\mathbb{H}$ with $\mathbb{R}$ and $\mathbb{C}$ respectively. However, this argument does not work for the 7-sphere, since the octonion numbers – the 8-dimensional analogue of $\mathbb{H}$ – do not form an associative algebra. This poses an interesting question: does this argument work for other dimensions?

Does it work for… let's say… the 2-dimensional sphere $\mathbb{S}^2$? Unfortunately, no, not really. In order for our argument to work for $\mathbb{S}^2$ we would have to find a 3-dimensional normed (associative) division algebra over $\mathbb{R}$. We will discuss this issue at the end of these notes, but for now it suffices to note that so such algebra exists. This hints at the fact that $\mathbb{S}^n$ *being a group* is a somewhat special characteristic of $n = 0, 1, 3$, but perhaps $\mathbb{S}^2$ admits some other *mysterious* group structure. This is not the case, however, since $\mathbb{S}^2$ is not parallelizable – and every Lie group is a parallelizable manifold.

Another interesting question one might ask is: what's the Lie algebra of $\mathbb{S}^3$? Well, by identifying a quaternion $q = a + bi + cj + dk$ with it's matrix representation

$$\mathbf{q} = \begin{pmatrix} a + bi & c + di \\ -c + di & a - bi \end{pmatrix} \in M_2(\mathbb{C})$$

we get a morphism of Lie groups $\Phi : \mathbb{H} \longrightarrow M_2(\mathbb{C})$ – see theorem 2.1.3 of [2]. Notice that $\mathbf{q}$ is a unitary matrix – i.e. $\mathbf{q} \in U(2)$. Moreover, $\det \mathbf{q} = a^2 + b^2 + c^2 + d^2 = |q|^2 = 1$.

This implies $\Phi(\mathbb{S}^3)$ is precisely SU(2). Now the kernel of the restriction $\Phi \restriction_{\mathbb{S}^3}$ is the set of q's such that

$$a + bi = a - bi = 1 \tag{1}$$

$$c + di = -c + di = 0 \tag{2}$$

In other words, $\ker \Phi \restriction_{\mathbb{S}^3} = \{1\}$ and so $\mathbb{S}^3 \cong \mathrm{SU}(2)$ as Lie groups. Hence the Lie algebra of $\mathbb{S}^3$ is

$$\mathfrak{su}_2 = \{M \in M_2(\mathbb{C}) : M = -M^*, \operatorname{Tr} M = 0\}$$

and, given that $\mathbb{S}^3$ is simply connected, all other Lie groups with coinciding Lie algebra are quotients of $\mathbb{S}^3$ by discrete subgroups of it's center – see the 8-th lecture of [5]. Such subgroups and their corresponding quotients of $\mathbb{S}^3$ will be *center* of the following discussion. However, as interesting as they may sound, this subgroups are quite scarce.

To see this, simply notice that the center of $\mathbb{H}$ is $\mathbb{R}$, from which follows that the center of $\mathbb{S}^3$ is $\mathbb{R} \cap \mathbb{S}^3 = \{1, -1\}$ – it has precisely one non-trivial subgroup, which is, of course, itself. Nevertheless, the quotient $\mathbb{S}^3/\{1,-1\}$ is quite an interesting example. Let's dive into it!

1 The Quotient of $\mathbb{S}^3$ by it's Center

We'll start this section by asking a simple question: who is the quotient of $\mathbb{S}^3$ by it's center? More precisely: what's it's geometric structure? Well, the action of 1 in the sphere is trivial, so we don't need to take it into account. All it's left to analyze is the action of -1, which isn't that complicated either: two elements p and q of $\mathbb{S}^3$ are in the same orbit under this action if $p = -q$. In other words, we're identifying diametrically opposed points of the 3-sphere.

Hence the quotient $\mathbb{S}^3/\{1,-1\}$ can be thought of the "upper semi-3-sphere". The concepts of "up", "down", "left" and "right" aren't precise in a 4-dimensional space, but the "upper semi-3-sphere" could be defined as… let's say… the set o unitary quaternions with non-negative real coefficients. One should point out that this isn't quite the case, since we're also identifying diametrically opposed *pure quaternions* – quaternions q with $\operatorname{Re} q = 0$ – but we'll leave such nuances for later.

Sharp-eyed observers may have realized that such space coincide with the quotient of the space of non-zero 4-dimensional vectors with real coefficients by the relation that identifies colinear points – which is to say, two vectors are considered the same if they are multiples of one-another – known as *the projective space* $\mathbb{R}\mathrm{P}^3$. Indeed, this is clearly the case: every line through the origin in $\mathbb{R}^4$ will intersect the 3-sphere in precisely two diametrically opposed points, so by choosing the point that lies in the "upper semi-3-sphere" we arrive at the desired conclusion.

This implies $\mathbb{R}\mathrm{P}^3 \cong \mathbb{S}^3/\{1,-1\}$ – at least as sets – but it also poses some interesting questions: who is $\mathbb{R}\mathrm{P}^3$? Why is it called *projective*? Why would anyone in God's green earth study such a smilingly arbitrary space? This are the questions we'll attempt to answer in this section, and we'll start by last of them. The short answer is that *no one actually studies such a quotient*: this is simply a construction of the abstract space $\mathbb{R}\mathrm{P}^3$, which is itself the thing that people study.

In general, the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$ naturally comes up when we search for a *somewhat Euclidean* space with the geometric property that *every pair of distinct straight lines intersect at a single point.* In other words, $\mathbb{R}\mathrm{P}^n$ is an "extension" of $\mathbb{R}^n$ such that each pair of lines in $\mathbb{R}^n$ can be naturally extended to a pair of *things* – subsets – with singleton intersection. This characterization is, in fact, the answer to our second question: $\mathbb{R}\mathrm{P}^n$ is called the *projective* because the image of two distinct 3-dimensional lines under a 2-dimensional *projection* always intersect at a single point – including parallel ones – as shown in figure 1.

This is quite an interesting characterization, but it is not at all clear that such a space exists. We've already said that *the quotient of the space of non-zero n-dimension vectors by the relation that identifies …* is a construction of $\mathbb{R}\mathrm{P}^n$, but how do we get from "two lines always have a single intersection point" to this? We'll start out by analyzing the case of the projective plane $\mathbb{R}\mathrm{P}^2$, mainly because we can draw 2-dimensional stuff. Notice, however, that all of the arguments used in following discussion work for arbitrary dimensions.

Figure 1: The Cologne Hauptbahnhof train station as seen from the central platform of the main hall. The projection of the lines at the boundary of the platform appear to intersect at a vanishing point in the horizon, even though they are parallel. Credit: Martin Falbisonel

If two lines in $\mathbb{R}^2$ are concurrent, then, by definition, they already intersect at some (unique) point. However, if two distinct lines are parallel to one another, no such an intersection point exists. Nevertheless, we could *add* this point – i.e. for each line that passes through the origin in $\mathbb{R}^2$ we add a point, which should could correspond to the intersection of this line with all other lines parallel to it.

Hence $\mathbb{R}\mathrm{P}^2$ can be informally thought-of as $\mathbb{R}^2 \cup \{\text{points at infinity}\}$ – where each "point at infinity" corresponds to a line through the origin. One way to turn our intuition into something practical is by considering the stereographic projection, which homeomorphically identifies the plane $\mathbb{R}^2$ with the upper 2-sphere $\mathbb{S}^2_+$:

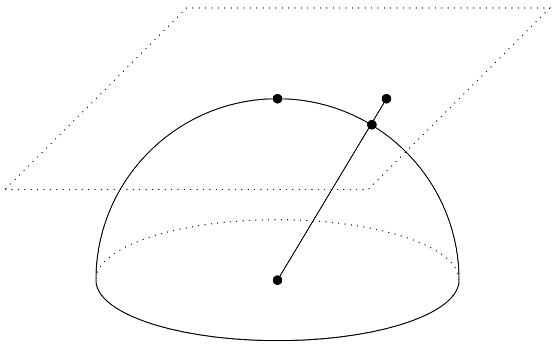

Figure 2: A graphical depiction of the stereographic projection: we map each point in the upper half of the sphere to the projection of this point in the tangent plane at the north pole by drawing a line between this point and the center of the sphere and then taking the intersection of this line with the plane.

Notice that the image of $p \in \mathbb{S}^2_+$ under the stereographic projection "explodes to infinity" as p approaches the boundary of $\mathbb{S}^2_+$. In fact, by taking spherical coordinates

$$\mathbb{S}^2_+ = \{(\cos\theta\cos\varphi, \sin\theta\cos\varphi, \sin\varphi) : 0 \leqslant \theta < \pi, 0 < \varphi < \pi\},$$

fixing θ and letting φ vary we can see that the image of the corresponding curve is the line $\gamma(t) = t(\cos\theta, \sin\theta)$ in $\mathbb{R}^2$ – where $t = \frac{1}{\tan\varphi}$. This can be shown explicitly by computing the formula of the stereographic projection in spherical coordinates, but we hope that figure 3 is sufficiently convincing. Moreover, the inverse image of lines that are parallel to $\gamma(t)$ in $\mathbb{R}^2$ are half (great) circles in $\mathbb{S}^2_+$ which starts at $(\cos\theta, \sin\theta, 0)$ and ends at $(-\cos\theta, -\sin\theta, 0)$, so that their intersection with $\gamma(t)$ has precisely two (diametrically opposed) points, both lying in the boundary of $\mathbb{S}^2_+$.

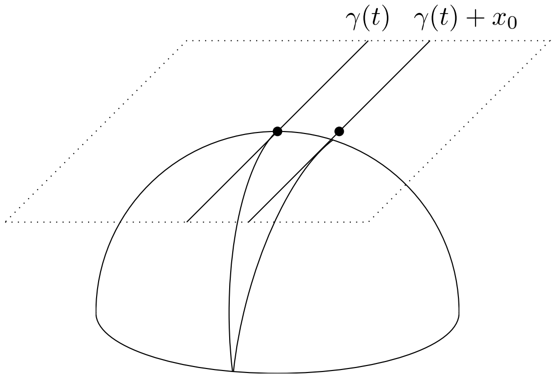

Figure 3: A graphical depiction of the projection of great circles in the plane: circles that pass trough the north pole are mapped to lines trough the origin, while other circles whose intersection with the equator match those of a given circle that passes trough the equator are mapped to parallel lines

In other words, by taking $\mathbb{R}\mathrm{P}^2 = \mathbb{S}^2_+ \cup \partial\mathbb{S}^2_+$ and identifying lines in $\mathbb{R}^2$ with half (great) circles in $\mathbb{S}^2_+$ we arrive at our desired construction – where the "points at infinity" are realized as the points in $\partial\mathbb{S}^2_+$. Although a bit cumbersome, this comes prepackaged with a topology which pretty much screams into the wind "I am the projective plane! I am an extension of $\mathbb{R}^2$ where all lines have a single intersection point!".

There's a small catch though: our previous derivations showed us each pair of parallel lines in $\mathbb{R}\mathrm{P}^2$ intersect at two diametrically opposed points in the boundary of $\mathbb{S}^2_+$, and we only want them to intersect at a single point. This is because by walking towards $\partial\mathbb{S}^2_+$ in $\gamma(t)$ at both ends we're essentially *approaching the same point at infinity from two different directions.* To fix this, we can simply identify diametrically opposed points in $\partial\mathbb{S}^2_+$.

In fact, we can simply identify all diametrically opposed points in $\mathbb{S}^2$ – or even identify colinear points in the entirety of $\mathbb{R}^3 \setminus \{0\}$, arriving at the usual construction of $\mathbb{R}\mathrm{P}^2$. As previously mentioned, nothing we've said so far is specific to $\mathbb{R}\mathrm{P}^2$. In particular, identifying diametrically opposed points in $\mathbb{S}^3$ is precisely the same as taking the quotient $\mathbb{S}^3/\{1, -1\}$. This hopefully establishes that $\mathbb{R}\mathrm{P}^3 \cong \mathbb{S}^3/\{1, -1\}$ not only because they coincide as sets, but because their geometry is the same.

2 Other Quotients

Even though subgroups of $\mathbb{S}^3$'s center are scarce, clearly these are not the only finite subgroups. For instance, given a positive integer n the subgroup generated by $e^{\frac{2i\pi}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is a cyclic group with n elements. In general, finite subgroups of $\mathbb{S}^3$ are called *finite rotation groups.* There's even a complete classification of such subgroups – see [4].

What's perhaps more surprising is that, even though the quotient $\mathbb{S}^3/G$ is *almost never* a Lie group for finite $G \subseteq \mathbb{S}^3$ – indeed, it usually isn't even a group – the following holds.

Theorem 2.1. *If $G \subseteq \mathbb{S}^3$ is a finite subgroup then the quotient space $\mathbb{S}^3/G$ is a 3-dimensional manifold.*

The fact that the orbit $Gq = \{gq : g \in G\}$ is finite implies we're identifying a very small amount of points in the quotient $\mathbb{S}^3/G$, which already hints at the fact that the quotient preserves much of the local structure of $\mathbb{S}^3$, but how do we prove it?

Proof. We'll start by showing that $\mathbb{S}^3/G$ is locally Euclidean. Given $q \in \mathbb{S}^3$, since $\mathbb{S}^3$ is locally Euclidean it suffices to find some neighborhood $U \subseteq \mathbb{S}^3/G$ of Gq such that $U \cong \pi^{-1}(U)$. Indeed, since $\mathbb{S}$ is locally Euclidean, showing such neighborhood exists is akin to showing that the quotient space is *locally locally Euclidean* – i.e. locally Euclidean. The obvious approach is to take some neighborhood $V \subseteq \mathbb{S}^3$ of q and try to prove that $U = \pi(V)$ is homeomorphic to V, but this clearly doesn't work for arbitrary V since the projection $\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^3/G$ is not injective.

We could, however, search for some V satisfying the above criteria such that $\pi \restriction_V$ is injective. To do so, we start by taking disjoint open neighborhoods $V_g \subseteq \mathbb{S}^3$ of gq for each $g \in G$. Now notice that for each $g \in G$, $g^{-1}V_g$ is an open neighborhood of q. Hence

$$V = \bigcap_{g \in G} g^{-1} V_g$$

is an open neighborhood of q.

We claim that $\pi \restriction_V$ is injective. Indeed, given distinct $g_1, g_2 \in G$ it follows from the fact that $g_1 V \subseteq V_{g_1}$ and $g_2 V \subseteq V_{g_2}$ that $g_1 V \cap g_2 V = \emptyset$. Hence if $v_1, v_2 \in V$ are such that $v_1 = g v_2$ for some $g \in G$ then $g = 1$ and therefore $v_1 = v_2$. In other words, if $G v_1 = G v_2$ then $v_1 = v_2$. This establishes that $\pi \restriction_V$ is injective. Now since the projection π is a surjective open function, $\pi : V \xrightarrow{\sim} \pi(V)$.

The proof that the quotient $\mathbb{S}^3/G$ is Hausdorff is quite similar to this: given two distinct points in the quotient, we use disjoint neighborhoods of each of their points to construct disjoint neighborhoods in the quotient space. Moreover, the fact that the quotient is second-countable follows immediately from the construction of the quotient topology: the projection of a basis for $\mathbb{S}^3$ is a basis for the quotient space. We are done. ■

Notice that almost nothing we've said so far is specific to $\mathbb{S}^3$. Indeed, the same argument could be adapted to a prove that…

Theorem 2.2. *If M is a n-dimensional manifold and G is a finite group acting continuously and freely on M – i.e. the actions of element of G other than the identity don't have any fixed points – then the quotient space M/G is an n-dimensional manifold.*

Proof. Just replace $\mathbb{S}^3$ with M and $q \in \mathbb{S}^3$ with $p \in M$ in the proof above. ■

In fact, theorem 2.1 can be though-of as a special case of an even more general result about Lie groups…

Theorem 2.3. *If M is a smooth manifold and G is a compact Lie group acting freely in M then the quotient space M/G is a smooth manifold.*

The proof of theorem 2.3 is way beyond the scope of this notes, but if you're interested in this please refer to theorem 2.33 of [1]. What's interesting about theorem 2.2 to us is that it allows us to look at quotients of other manifolds by continuous actions of $\mathbb{S}^3$. An interesting example of such actions is the action of $\mathbb{S}^3$ in $\mathbb{H}$ under conjugation.

First of all, notice the set set of pure quaternions $\{p \in \mathbb{H} : \operatorname{Re} p = 0\}$ is stable under this action. So $\mathbb{S}^3$ acts on the set of pure quaternions. Moreover, the map $p \longmapsto qpq^{-1}$ is clearly linear and $\|qpq^{-1}\| = \|q\| \cdot \|p\| \cdot \|q^{-1}\| = \|p\|$. Hence by fixing the basis $\{i, j, k\}$ and identifying the space of pure quaternions with $\mathbb{R}^3$ we arrive at the infamous *adjoint action*

$$\begin{aligned} \operatorname{Ad} : \mathbb{S}^3 &\longrightarrow \operatorname{SO}(3) \\ q &\longmapsto \operatorname{Ad}(q) : \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ &\qquad\qquad\qquad p \longmapsto qpq^{-1} \end{aligned}$$

As simple as it is, this description of the adjoint action is a bit lacking in the sense that it relies in the identification of $\mathbb{R}^3$ with the set of pure quaternions. Ideally we would like to describe $\operatorname{Ad}(q)$ just in terms of it's matrix representation.

For instance, by computing

$$\mathrm{Ad}(e^{i\theta})\ i = (\cos\theta + i\sin\theta)i(\cos\theta - i\sin\theta) = i \tag{3}$$
$$\mathrm{Ad}(e^{i\theta})\ j = (\cos\theta + i\sin\theta)j(\cos\theta - i\sin\theta) = j\cos 2\theta + k\sin 2\theta \tag{4}$$
$$\mathrm{Ad}(e^{i\theta})\ k = (\cos\theta + i\sin\theta)k(\cos\theta - i\sin\theta) = -j\sin 2\theta + k\cos 2\theta \tag{5}$$

we can see that the matrix representation of $\mathrm{Ad}(e^{i\theta})$ in the basis $\{i, j, k\}$ is

$$\mathrm{Ad}(e^{i\theta}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix},$$

i.e. a rotation by 2θ radians around the x-axis. We claim that, given $p \in \mathbb{S}^3$ with $\mathrm{Re}\, p = 0$ and $p \neq -i$, $\mathrm{Ad}(\cos\theta + p\sin\theta)$ acts on $\mathbb{R}^3$ as the rotation by 2θ radians around the p-axis – see exercise 2.4.7 of [2] – as shown in figure 4.

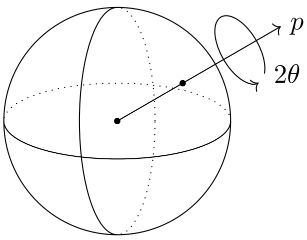

Figure 4: A sphere crossed by a line through the origin labeled as “p”. The coordinates of p induce a line through the origin, and conjugation by $\cos\theta + p\sin\theta$ acts as rotation by 2θ around this axis.

Of course, this is not a free action of a finite group, but it can be used to find interesting examples of those. For instance, the action of the subgroup generate by $\cos\frac{\pi}{n} + k\sin\frac{\pi}{n}$ in $\mathbb{S}^2$ identifies meridional sections of the sphere, as shown in figure 5. Now be removing the poles – i.e. the only points fixed by any of the elements of our subgroup – we get a free action and we see that it’s quotient is indeed a 2-dimensional manifold homeomorphic to a cylinder with no caps.

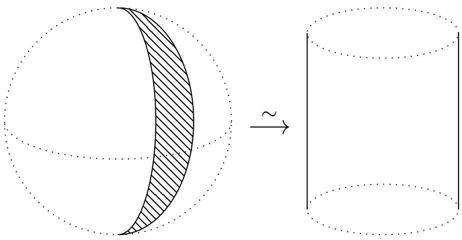

Figure 5: A sphere with the region between two distinct meridians highlighted, followed by an arrow pointing to a cylinder. The generator of the subgroup acts by rotating a point on the sphere by $\frac{2\pi}{n}$ around the z-axis, so the quotient space is the same as a meridian section of the sphere without the poles and with the meridians themselves identified. By dilating the circles parallel to the equator we can morph such quotient into a rectangle where vertical edges are identified, which is a cylinder with no caps.

What’s perhaps more amusing is that, since the image of the adjoint action contains all rotations, it spans the entirety of $\mathrm{SO}(3)$. Now notice that $\ker \mathrm{Ad}$ is precisely $Z(\mathbb{H}) \cap \mathbb{S}^3$, also known as $Z(\mathbb{S}^3) = \{1, -1\}$ – i.e. Ad is *almost faithful*. In other words, the unitary quaternions $\mathbb{S}^3$ *are* the orientation-preserving isometries of $\mathbb{R}^3$, at least as long as we’re willing to ignore their signs. This is quite useful for graphical computing: we’ve now reduced the problem of representing 3×3 matrices to that of representing 4-dimensional vectors, and we’re reduced the problem of computing and composing rotations to that of multiplying quaternion numbers – which is much faster to do.

Even more so, this is also a really interesting intuition for the quaternions themselves. As Willian Rowan Hamilton – the guy who discovered the quaternion numbers – wrote, “I am never satisfied unless I think that I can look beyond or through the signs to the things signified” [3] – i.e. this sort of intuitions are what actually endow what we study with meaning. Inspired the (at the time) recently discovered geometric intuition of the complex plane, Hamilton himself was interested in hypercomplex numbers related to the 3-dimensional space $\mathbb{R}^3$, and he spend the better part of 1843 trying to achieve this.

He failed, however, and for a good reason: Frobenius would later show that every finite-dimensional division over $\mathbb{R}$ is one of $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ – in particular, there is no 3-dimensional division algebra over $\mathbb{R}$. Nevertheless, Hamilton later realised that the solution lied in the introduction of a forth coordinate: “I then and there felt the galvanic circuit of though *close*; and the sparks which fell from it were the *fundamental equations*”. According to himself, Hamilton could not “resit the impulse – unphilosophical as it may have been – to cut (the quaternion formula) with a knife on a stone” on a nearby bridge.

If you’re interested in Hamilton’s metaphysical considerations on quaternion numbers, space & time I strongly recommend reading the historical interlude of the 4-th chapter of *Introduction to Representation Theory* [3], but hopefully what we’ve said so far establishes that the action of the 3-sphere on $\mathbb{R}^3$ is *fundamental* to the understanding of the quaternions themselves. In other words, the fact that the group structure of $\mathbb{S}^3$ comes from $\mathbb{H}$ is not a coincidence: in some sense, it is the reason why $\mathbb{H}$ was first discovered.

This provides further evidence for our claim that *the fact that* $\mathbb{S}^3$ – the main theme of the preceding discussion – *is a group is something very particular of* $n = 3$. We conclude these notes by quoting the words of a plaque that now marks the site of Hamilton’s vandalism:

> Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genious discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ & cut it on a stone of this bridge.

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