$\mathcal{VB}$-Groupoids and the category of multiplicative sections

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Abstract

We define the categories of 2-Vector Spaces and 2-Term complexes and show that they are equivalent. Then, we define $\mathcal{VB}$-groupoids and explore the concept of representation up to homotopy. In the last part we define what a multiplicative section of a $\mathcal{VB}$-groupoid is and show that the category of multiplicative sections of a $\mathcal{VB}$-groupoid is isomorphic to a 2-vector space.

1 Introduction

This work has the objective of giving an overview of $\mathcal{VB}$-groupoids and to show a proof of the fact that the collection of multiplicative sections $\mathcal{VB}$-groupoid has the strucutre of a category that is isomorphic to a 2-Vector space. Out work has the following structure:

- In the first section, we define internal categories and internal functors in order to define 2-Vector spaces. Then, we define the category of 2-Term Chain Complexes and describe an equivalence between them. In this section, we follow [BC03] closely.
- In the second section, we start by recalling some notions of Lie groupoid theory that are used on the subsequent text. Then, we define $\mathcal{VB}$-groupoids, explore some examples and the concept of representations up to homotopy of a Lie groupoid, mentioning how a representation up to homotopy allows us to "decompose´´ $\mathcal{VB}$-groupoids in a convenient way. For this section, we follow closely our main reference [OW19], and [GSM10].
- In this final section, we define what a multiplicative section of a $\mathcal{VB}$-groupoid is, prove that they can be given the structure of a category. We also give conditions to characterize its morphisms as certain sections of a vector bundle and vice-versa, and show that this category is isomorphic to a 2-Vector space. In this part we follow [OW19] very closely.

2 2-Vector spaces

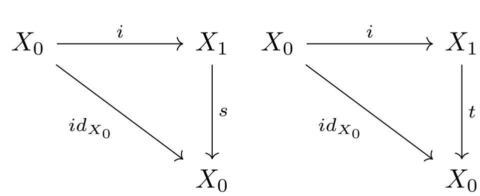
Let us recall the definition of an internal category.

Definition 2.1. Given a category X with pullbacks, an **internal category** K in X consists of:

- an object X_0 of objects;
- an object X_1 of arrows;

and morphisms $s, t : X_1 \rightrightarrows X_0$ (**source** and **target** morphisms), $i : X_0 \to X_1$ (**identity-assigning** morphisms), $m : X_1 \times_{X_0} X_1 \to X_1$(**composition** morphisms) satisfying the following commutativity conditions:

1. on diagrams specifying the source and target of identity morphisms:



2. on diagrams specifying the source and target of compositions:

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{p_1} & X_1 \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle s} \\ X_1 & \xrightarrow{s} & X_0 \end{array} \qquad \begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{p_2} & X_1 \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle t} \\ X_1 & \xrightarrow{t} & X_0 \end{array}$$

3. on the diagram expressing the associativity of compositions

$$\begin{array}{ccc} X_1 \times_{X_0} \times X_1 \times_{X_0} \times X_1 & \xrightarrow{m \times id_{X_1}} & X_1 \times_{X_0} X_1 \\ \downarrow{\scriptstyle id_{X_1} \times m} & & \downarrow{\scriptstyle m} \\ X_1 \times_{X_0} X_1 & \xrightarrow{m} & X_1 \end{array}$$

4. on the diagram specifying the right and left unit laws

$$\begin{array}{ccccc} X_0 \times_{X_0} X_1 & \xrightarrow{i \times_{X_0} id_{X_1}} & X_1 \times_{X_0} X_1 & \xleftarrow{id_{X_1} \times_{X_0} \times i} & X_1 \times_{X_0} X_0 \\ & \searrow{\scriptstyle p_2} & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle p_1} & \\ & & X_1 & & \end{array}$$

where the pullback $X_1 \times_{X_0} X_1$ defined is of the source and target morphisms.

Now we internalize the morphisms of categories:

Definition 2.2. Given internal categories X, X', an **internal functor** $F : X \to X'$ consists of two morphisms, $F_0 : X_0 \to X'_0$ and $F_1 : X_1 \to X'_1$ such that the following commutativity conditions are satisfied:

1. on the diagram expressing the preservation of source and target morphisms

$$\begin{array}{ccc} X_1 & \xrightarrow{s} & X_0 \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ X'_1 & \xrightarrow{s'} & X'_0 \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{t} & X_0 \\ \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} \\ X'_1 & \xrightarrow{t'} & X'_0 \end{array}$$

2. on the diagram that specifies that functors preserve identities

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X_1 \\ \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle F_1} \\ X'_0 & \xrightarrow{i'} & X'_1 \end{array}$$

3. on the diagram that expresses that compositions are preserved by functors

$$\begin{array}{ccc} X_1 \times_{X_0} X_1 & \xrightarrow{F_1 \times F_1} & X'_1 \times_{X'_0} X_1 \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m'} \\ X_1 & \xrightarrow{F'_1} & X'_1 \end{array}$$

Internal categories and internal functors are concepts that express the idea of "an "object" inside a given category that behaves like one, respecting the structure of objects and morphisms of first category".

Definition 2.3. A 2**-vector space** is an internal category on the category of Real Vector Spaces and Linear maps $\mathrm{Vect}_{\mathbb{R}}$.

In explicit terms, a 2-Vector Space consists of two **vector spaces** V_0 and V_1, where V_0 is the vector space of objects, and V_1 is the vector space of arrows; together with source, target, identity-assigning and composition **linear maps** as defined on Definition 2.1.

One useful observation for 2-Vector spaces is that composition on V_1 can be expressed using the vector space operations of V_1: we rewrite morphisms $f : x \to y \in V_1$ as $(x, \vec{f}) : 0 \to y - x$, where $\vec{f} = f - i(s(f))$ is its **arrow part**, and composition takes the form $(y, \vec{g})(x, \vec{f}) := (x, \vec{g} + \vec{f})$. A proof of this fact can be found on [BC03]

Definition 2.4. Given 2-Vector spaces V and W, a **linear functor** from V to W is an internal functor $F : V \to W$. 2-Vector Spaces and linear functors form a category, called 2**-Vect**.

We describe now an equivalence of 2-Vect with the category 2**-Term** of 2**-term chain complexes**, that is, pairs of real vector spaces with a linear map between them

$$W_1 \xrightarrow{d} W_0;$$

as morphisms we have **chain maps** between them, i.e., pairs $\varphi_1 : W_1 \to W_1', \varphi_0 : W_0 \to W_0'$ of linear maps such that the following diagram commutes:

$$\begin{array}{ccc} W_1 & \xrightarrow{d} & W_0 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} \\ W_1' & \xrightarrow{d'} & W_0'. \end{array}$$

To describe an equivalence, we define two functors $T : \text{2-Vect} \to \text{2-Term}$ and $S : \text{2-Term} \to \text{2-Vect}$ whose compositions are naturally isomorphic to the identity functors.

- *Definition of* T: on objects, T takes a 2-Vector Space $V_1 \rightrightarrows V_0$ and maps it to the 2-Term chain complex given by

$$\ker(s) \xrightarrow{t|_{\ker(s)}} V_0.$$

  On morphisms, T takes a linear functor from $V_1 \rightrightarrows V_0$ to $V_1' \rightrightarrows V_0'$ given by (F_0, F_1) from and maps it to the chain map $\varphi_0 := F_0 : V_0 \to V_0'$, $\varphi_1 := F_1 |_{\ker(s)}: \ker(s) \to \ker(s')$. Note that φ_1 is well-defined since for $(x, \vec{f}) \in \ker(s)$, $s' \circ F_1(x, \vec{f}) = F_0 \circ s(x, \vec{f}) = 0$, and since F_1 commutes with t, it is a chain map.

- *Definition of* S: on objects, S takes a 2-Term Chain Complex $W_1 \xrightarrow{d} W_0$ and maps it into the 2-Vector Space $W_0 \oplus W_1 \rightrightarrows W_0$ with source, target and identity-assigning maps given by $s(x, \vec{f}) = x$, $t(x, \vec{f}) = x + d\vec{f}$ and $i(x) = (x, 0)$. Note that $W_1 \cong \ker(s)$, and the decomposition $W_0 \oplus W_1$ is precisely the decomposition of morphisms into sources and arrow parts.

  On morphisms, S takes a chain map (φ_0, φ_1) and maps into the linear functor defined as $F_0 := \varphi_0$ and $F_1 = \varphi_0 \oplus \varphi_1$. Note that F is linear on objects and morphisms and preserves the source, target, identity and composition maps, since they are defined in terms of addition and the differential in the chain complexes, and both φ_0 and φ_1 are linear and preserve the differential.

The natural isomorphisms $\gamma : TS \implies 1_{\text{2-Vect}}$ and $\delta : ST \implies 1_{\text{2-Term}}$ are defined as follows:

- *Definition of* γ: Since TS maps a 2-Vector Space $V_1 \rightrightarrows V_0$ (which we refer to as V) into a 2-Vector Space $V_0 \oplus \ker(s) \rightrightarrows V_0$ (which we refer to as V'), with s, t the source and target maps of $V_1 \rightrightarrows V_0$ and $s'(x, \vec{f}) = x$, $t'(x, \vec{f}) = x + t\vec{f}$ the source and target maps of $V_0 \oplus \ker(s) \rightrightarrows V_0$.

  We can define then an isomorphism γ_V by

$$(\gamma_V)_0 : x \in V_0 \mapsto x \in V_0,$$

$$(\gamma_V)_1 : (x, \vec{f}) \in V_0 \oplus \ker(s) \mapsto i(x) + \vec{f} \in V_1$$

  The fact that γ_V is as isomorphism follows from the uniqueness of the decomposition of $f \in V_1$ as $(x, \vec{f})$.

- *Definition of* δ: Since ST maps a 2-Term chain complex $W_1 \xrightarrow{d} W_0$ to a 2-Term Chain Complex given by $\ker(s) \xrightarrow{t|_{\ker(s)}} W_0$, where s, t are respectively the source and target maps of the 2-Vector Space $W_0 \oplus W_1 \rightrightarrows W_0$ given by $s(x, \vec{f}) = x$ and $t(x, \vec{f}) = x + d\vec{f}$, we define δ as the chain map (φ_0, φ_1) where $\varphi_0 : W_0 \to W_0$ is the identity and $\varphi_1 : \ker(s) \to W_1$ is given by the isomorphism $(0, \vec{f}) \mapsto \vec{f}$.

From this discussion we can conclude that

Theorem 2.1. The categories **2-Vect** and **2-Term** are equivalent.

3 $\mathcal{VB}$-groupoids

We recall here basic concepts relating groupoids and Lie groupoids. A reference for this content is [CF11].

Definition 3.1. A **groupoid** $\mathcal{G} \rightrightarrows M$ consists of two sets, the **set of arrows** $\mathcal{G}$, the **set of objects** M, together with:

- **source** and **target** maps $s, t : \mathcal{G} \to M$ mapping each arrow g to its source $s(g)$ and target $t(g)$;
- a **composition map** m defined on the set $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G}^2 : s(g) = t(h)\}$ that takes two composable arrows to a new arrow $m(g, h) := gh$ in $\mathcal{G}$;
- a **unit map** $1 : M \to \mathcal{G}$ that maps elements m in M to identity arrows 1_m in $\mathcal{G}$;
- a **inverse map** $i : \mathcal{G} \to \mathcal{G}$ that takes arrows g to their inverses g^{-1};

where those maps (which we call **structure maps**) satisfy the following:

1. for every $(g, h) \in \mathcal{G}^2$, $s(gh) = s(h)$ and $t(gh) = t(g)$;
2. for every $(g, h), (h, k) \in \mathcal{G}^2$, then $(gh)k = g(hk)$;
3. for every $f, g \in \mathcal{G}$ and $x \in M$ such that $(f, 1_x), (1_x, g) \in \mathcal{G}^{(2)}$, $f1_x = f$ and $1_x g = g$;
4. for every $g \in \mathcal{G}$, $g^{-1}g = 1_{s(g)}$ and $gg^{-1} = 1_{t(g)}$.

Using categorical language, a groupoid is just a small category where every morphism is invertible.

Definition 3.2. A **Lie groupoid** is a groupoid $\mathcal{G} \rightrightarrows M$ where both the set of objects and arrows are manifolds, all structure maps are smooth and the source and target maps are submersions.

Example 3.1. A Lie group G is a Lie groupoid $G \rightrightarrows \{*\}$ over the trivial manifold.

Example 3.2. Given a vector bundle $\pi : E \to M$, there is a Lie groupoid $GL(E) \rightrightarrows M$ with M as the set of objects and morphisms $f : x \to y \in GL(E)$ are isomorphisms between the fibers of E_x and E_y.

Example 3.3. Given a groupoid $\mathcal{G} \rightrightarrows M$, a manifold N and a smooth map $f : N \to M$, we define a **left groupoid action along** f to be a smooth map $\theta : G \times_M N \to N$ such that the following hold:

1. $f(\theta_g(x)) = t(g)$, for x in N;
2. if $(g, h) \in \mathcal{G}^{(2)}$, $\theta_g(\theta_h(x)) = \theta_{gh}(x)$
3. $\theta_{1_{(x)}} = \mathrm{id}\,|_{f^{-1}(x)}$

Given a left groupoid action of $\mathcal{G} \rightrightarrows M$ along $f : N \to M$, we can define an **action groupoid** $G \ltimes N \rightrightarrows N$ with N as its set of objects, and as arrows we have pairs $(g, x) \in \mathcal{G} \times_M N$ with source $s(g, x) = g$, target $t(g, x) = \theta_g(x)$ and composition given by $(g, y)(h, x) = (gh, x)$. In the case that $G \rightrightarrows \{*\}$ is a Lie group, the only possible smooth map $f : N \to \{*\}$ is the constant one, so $G \times_m N = G \times N$ and we regain the notion of a left action of a Lie group G on a manifold N. The corresponding action groupoid has $G \times N$ as set of objects. The notion of right action along a map and its corresponding action groupoid are defined analogously.

Definition 3.3. Given two Lie Groupoids $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$, a **morphism of Lie groupoids** is a pair of maps (F_0, F_1) where $F_0 : M \to N$, $F_1 : \mathcal{G} \to \mathcal{H}$ preserve the groupoid structure, i.e., they satisfy the following:

1. if $g : x \to y$, then $F_1(g) : F_0(x) \to F_0(y)$;
2. if $(g, h) \in \mathcal{G}^{(2)}$, $F_1(gh) = F_1(g)F_1(h)$;
3. if $x \in M$, $F_1(1_x) = 1_{F_0(x)}$.

Using categorical language, a morphism of Lie groupoids is simply a functor.

Definition 3.4. A **representation** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of a vector bundle $\pi : E \to M$ and a morphism $F : \mathcal{G} \to GL(E)$.

We recall here briefly the definition of the Lie algebroid of a Lie groupoid. It will appear as an example in the next section.

Definition 3.5. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The **Lie algebroid of** $\mathcal{G}$ is the vector bundle $\mathrm{Lie}(\mathcal{G}) := M \times_{\mathcal{G}} \ker(ds)$ with the restriction of the vector bundle map $dt\,|_{\ker(\mathrm{ds})}$ as its **anchor map**, and a Lie bracket $[\,\cdot\,;\cdot\,]_{\mathrm{Lie}(\mathcal{G})}$ on its space of sections that is induced by the Lie bracket on the space of right-invariant vector fields on $\mathcal{G}$.

3.1 $\mathcal{VB}$-groupoids

In the study of Manifolds, vector bundles are important objects. Many important objects are defined using them: from vector fields to differential forms, connections, and so on. What we introduce now is the corresponding notion of vector bundle for a Lie groupoid $\mathcal{G} \rightrightarrows M$. Since a vector bundle over a manifold M is a manifold that is a union of vector spaces that satisfy a certain local triviality condition, it is natural that a "vector bundle groupoid" over a Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of a Lie groupoid where both sets satisfy a local triviality condition over the corresponding arrow/object sets, such that the groupoid and vector bundle structures are compatible. We formalize this discussion in the following definition:

Definition 3.6. A **$\mathcal{VB}$-groupoid** over a Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of a Lie groupoid $\mathcal{V} \rightrightarrows E$ together with surjective submersions $q_{\mathcal{V}} : \mathcal{V} \to \mathcal{G}$ and $q_E : E \to M$ making both into vector bundles which are compatible with the Lie groupoid structure, i.e., the structure maps $\tilde{s}, \tilde{t}, \tilde{1}, \tilde{i}, \tilde{m}$ of $\mathcal{V} \rightrightarrows E$ are vector bundle morphisms over the corresponding structure maps of $\mathcal{G} \rightrightarrows M$.

We represent a $\mathcal{VB}$-groupoid as a square:

$$\begin{array}{ccc} \mathcal{V} & \overset{\tilde{s}}{\underset{\tilde{t}}{\rightrightarrows}} & E \\ {\scriptstyle q_{\mathcal{V}}}\downarrow & & \downarrow{\scriptstyle q_E} \\ \mathcal{G} & \overset{s}{\underset{t}{\rightrightarrows}} & M. \end{array}$$

It follows from the definition that $(q_{\mathcal{V}}, q_E)$ is a morphism of Lie groupoids from $\mathcal{V} \rightrightarrows E$ to $\mathcal{G} \rightrightarrows M$.

Definition 3.7. We define the **core** of $\mathcal{V} \rightrightarrows E$ to be the pullback $C := M \times_{\mathcal{G}} \ker(\tilde{s})$ of $1 : M \to \mathcal{G}$ and $q_{\mathcal{V}} |_{\ker(\tilde{s})}: \ker(\tilde{s}) \to \mathcal{G}$, that is, the arrows v_{1_x} of $\mathcal{V}$ that are in $\ker(\tilde{s})$.

Since $\tilde{s}$ is a surjective submersion, it has constant rank and thus $\ker(\tilde{s})$ is well-defined as a vector bundle over $\mathcal{G}$.

Proposition 3.1. Let $\mathcal{V} \rightrightarrows E$ be a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$, and let C be its core. Then the following sequence is an exact sequence of vector bundles over $\mathcal{G}$:

$$0 \longrightarrow t^*C \xrightarrow{\ r\ } \mathcal{V} \xrightarrow{\ q\ } s^*E \longrightarrow 0$$

where $r(c_{t(g)}) := c_{t(g)}\tilde{0}_g$, and $q(v_g) = (g, \tilde{s}(v))$.

Proof: Let $(g_1, g_2) \in \mathcal{G}^{(2)}$. The fact that $\tilde{m}$ is a vector bundle morphism implies that right-multiplication by g_2 induces a linear isomorphism on the fibers $\ker(\tilde{s})_{g_1}$ with $\ker(\tilde{s})_{g_1 g_2}$. In particular, it induces a linear isomorphism from $\ker(\tilde{s})_{1_{t(g)}}$ to $\ker(\tilde{s})_g$. However, $\ker(q) = \ker(\tilde{s})$, hence we have a vector bundle isomorphism from t^*C to $\ker(q)$, which implies the exactness of the sequence. $\square$

Some examples of $\mathcal{VB}$-groupoids follow:

Example 3.4. A $\mathcal{VB}$-groupoid over the trivial Lie Groupoid $\{*\} \rightrightarrows \{*\}$ is simply a 2-Vector Space, since a vector bundle over a point is just a vector space.

Example 3.5. If $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, application of the Tangent functor gives us the **Tangent groupoid** $T\mathcal{G} \rightrightarrows TM$. It is a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$ with core $\mathrm{Lie}(\mathcal{G})$.

Example 3.6. Dualizing the previous example, we obtain the **cotangent groupoid** $T^*\mathcal{G} \rightrightarrows \mathrm{Lie}(\mathcal{G})^*$, with core T^*M. It is a particular case of a **dual of a $\mathcal{VB}$-groupoid**: Given a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ with core C, the dual $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$ is $\mathcal{V}^* \rightrightarrows C^*$. For a detailed construction of its strucutre maps, see [Mac05].

Example 3.7. Given a representation of a Lie groupoid $F : \mathcal{G} \to GL(E)$, the action groupoid $G \ltimes E = s^*E \rightrightarrows E$ has a structure of $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$ with zero core since $C := M \times_{\mathcal{G}} \ker(\tilde{s}) = \{v_{1_m} \in \ker(\tilde{s}) : \tilde{s}(v_{1_m}) = 0_m\}$ and $\tilde{s}$ is the coordinate projection on fibers, thus $\ker(\tilde{s}) = \{(g, 0_g) \in \mathcal{G} \times E\}$ and C consists of the zero section on the units of $\mathcal{G}$. In fact, as a consequence of the exactness of the core sequence, every $\mathcal{VB}$-groupoid with zero core is of this form.

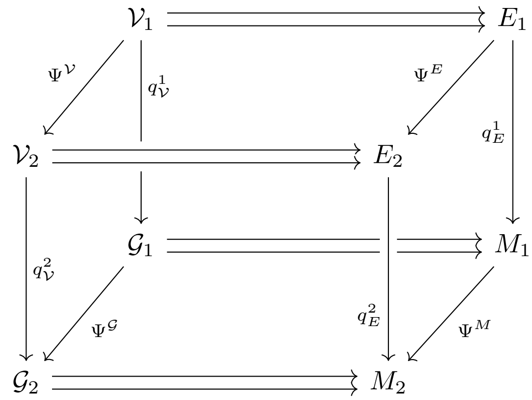
Example 3.8. Recall that an **G-equivariant vector bundle** consists of a vector bundle $\pi : E \to M$ together with an action of a Lie group G on E and M such that E and M are left G-spaces, the projection is G-equivariant (i.e. $\pi(gx) = g\pi(x)$) and left-multiplication $l_g : E_x \to E_{gx}$ is linear.

G-equivariant vector bundles are equivalent to representations of the groupoid $G \ltimes M \rightrightarrows M$. Indeed, given a representation F of $G \ltimes M \rightrightarrows M$ on a vector bundle E, we can define a G-action on E by $(g, e_m) \mapsto F_g(e_m)$; and conversely given a G-equivariant vector bundle, we can define a representation by $(g, m) \mapsto l_g(e_m)$ on arrows and by the identity on objects.

Consequently, given a G-equivariant vector bundle $E \to M$, the groupoid $s^*E \rightrightarrows E$ has a $\mathcal{VB}$-groupoid structure over $G \ltimes M \rightrightarrows M$. However, $s^*E = \{((g, m), e_n) \in (G \ltimes M) \times E : m = n\} = \{(g, e_m) \in G \times E\} = G \ltimes E$, so $G \ltimes E \rightrightarrows E$ is a $\mathcal{VB}$-groupoid over $G \ltimes M \rightrightarrows M$.

We now introduce the concept of *representation up to homotopy* of a Lie groupoid. It is a generalization of the notion of representation of a Lie groupoid, and will be used to obtain a useful decomposition of a $\mathcal{VB}$-groupoid that will allow us to make explicit computations later. For that, we need some definitions first:

Definition 3.8. Let $\mathcal{V}_1 \rightrightarrows E_1$ and $\mathcal{V}_2 \rightrightarrows E_2$ be two $\mathcal{VB}$-groupoids over $\mathcal{G}_1 \rightrightarrows M_1$ and $\mathcal{G}_2 \rightrightarrows M_2$ respectively. A **$\mathcal{VB}$-morphism** is a quadruple $(\Psi^{\mathcal{V}}, \Psi^E, \Psi^{\mathcal{G}}, \Psi^M)$ where $(\Psi^{\mathcal{V}}, \Psi^E)$ and $(\Psi^{\mathcal{G}}, \Psi^M)$ are Lie groupoid morphisms, $(\Psi^{\mathcal{V}}, \Psi^{\mathcal{G}})$ and (Ψ^E, Ψ^M) are vector bundle morphisms and the following diagram commutes:



Definition 3.9. A **quasi-action** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ on a vector bundle $E \to M$ is a smooth map $\Delta : \mathcal{G} \to GL(E)$ such that $\Delta_g : E_{s(g)} \to E_{t(g)}$ for every $g \in G$. A quasi-action is **unital** if $\Delta_{1_x} = Id$, for every $x \in M$; and **flat** if $\Delta_{gh} = \Delta_g \Delta_h$ for composable pairs $(g, h) \in \mathcal{G}^{(2)}$.

Using these concepts, a representation of a lie groupoid on a vector bundle is just a flat unital quasi-action.

Definition 3.10. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. A **representation up to homotopy** of $\mathcal{G}$ on the graded vector bundle $C \oplus E$ is given by a quadruple $(\partial, \Delta^C, \Delta^E, \Omega)$ where:

- $\partial : C \to E$ is a bundle map;
- Δ^C and Δ^E are unital quasi-actions on C and E respectively;
- $\Omega \in \mathrm{Hom}(s^*E, t^*C)$ is a section assigning to each composable pair $(g_1, g_2) \in \mathcal{G}^{(2)}$ is a linear map Ω_{g_1,g_2} which is **normalized**, i.e., $\Omega_{g_1,g_2} = 0$ if either g_1 or g_2 is a unit, satisfying the following conditions:

$$\Delta^E_{g_1} \circ \partial = \partial \circ \Delta^C_{g_1}$$
$$\Delta^C_{g_1} \Delta^C_{g_2} - \Delta^C_{g_1 g_2} + \Omega_{g_1,g_2} \partial = 0$$
$$\Delta^E_{g_1} \Delta^E_{g_2} - \Delta^E_{g_1 g_2} + \partial \Omega_{g_1,g_2} = 0$$
$$\Delta^C_{g_1} \Omega_{g_2,g_3} - \Omega_{g_1 g_2,g_3} + \Omega_{g_1,g_2 g_3} + \Omega_{g_1,g_2} \Delta^E_{g_3} = 0$$

for every composable triple (g_1, g_2, g_3).

As explained in [GSM10], such a structure induces a Lie groupoid structure on the vector bundle $t^*C \oplus s^*E \to \mathcal{G}$ turning it into a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$ with core C and having E as set of objects. Its structure maps are defined as follows:

- $\tilde{s}(c, g, e) = e$,
- $\tilde{t}(c, g, e) = \partial(c) + \Delta^E_g(e)$,
- $\tilde{1}_e = (0^C(x), 1(x), e), e \in E_x$,
- $(c_1, g_1, e_1)(c_2, g_2, e_2) = (c_1 + \Delta^C_{g_1}(c_2) - \Omega_{g_1,g_2}(e_2), g_1 g_2, e_2)$,
- $(c, g, e)^{-1} = (-\Delta^C_{g^{-1}}(c) + \Omega_{g^{-1},g}(e), g^{-1}, \partial(c) + \Delta^E_g(e))$.

A $\mathcal{VB}$-groupoid of the form $t^*C \oplus s^*E \rightrightarrows E$ together with the structure maps just defined is a **split** $\mathcal{VB}$-groupoid.

In the opposite way, a representation up to homotopy can be obtained from a $\mathcal{VB}$-groupoid. First, note that for each $x \in M$, the fiber $\mathcal{V}_x$ is canonically isomorphic to $C_x \oplus E_x$ by $(c, e) \mapsto c + \tilde{1}(e)$, in an analogous manner to the isomorphism described in the discussion preceding Theorem 2.1.

Now, an **horizontal lift** of $\mathcal{V}$ is a map $h : s^*E \to \mathcal{V}$ such that $q \circ h = 1_E$ and at every unit $x \in M$ it coincides with the decomposition just described. In [GSM10] it is proved that horizontal lifts always exist (albeit not in a canonical way, their existence relies on a partition of unity argument), and are used to define a representation up to homotopy of $\mathcal{G} \rightrightarrows M$. Even more can be said, as it is proved that the choice of a horizontal lift induces a $\mathcal{VB}$-isomorphism of a $\mathcal{VB}$-groupoid with a split $\mathcal{VB}$-groupoid.

We will use this isomorphism to study the structure of multiplicative sections - the "correct" notion of sections for $\mathcal{VB}$-groupoids. For that, we will need expressions for the right and left-invariant section of a split $\mathcal{VB}$-groupoid. By the definition of the structure maps, we have:

- $c^r(g) := c_{t(g)} \cdot \tilde{0}_g = (c_{t(g)}, g, 0)$;
- $c^l(g) = -\tilde{0}_g \cdot \tilde{i}(c_{s(g)}) = -\tilde{0}_g \cdot (c_{s(g)}, 1_{s(g)}, \partial(c)) = (\Delta^C_g(c_{s(g)}, g, -\partial(c_{s(g)}))$.

4 The category of multiplicative sections

When vector bundles are studied, an object of particular interest are sections. In this section, we will define what should be a section of a $\mathcal{VB}$-groupoid and show that they have not only the structure of a vector space, but that of a 2-Vector space.

Since a $\mathcal{VB}$-groupoid consists not only of two vector bundles, but two Lie groupoid structures on the total and base spaces, the correct concept of section of a $\mathcal{VB}$-groupoid must respect the groupoid structure, i.e., it should be a morphism of Lie groupoids.

Definition 4.1. Let $\mathcal{V} \rightrightarrows E$ be a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$. A **multiplicative section** is a pair $(V, e) \in \Gamma(\mathcal{V}) \oplus \Gamma(E)$ that is a morphism of Lie groupoids from $\mathcal{G} \rightrightarrows M$ to $\mathcal{V} \rightrightarrows E$.

The set of all multiplicative sections is denoted $\Gamma_{\text{mult}}(\mathcal{V})$. It can be made into a vector subspace of $\Gamma(\mathcal{V}) \oplus \Gamma(E)$ since the structure maps are fiberwise linear.

Example 4.1. If $V_1 \rightrightarrows V_0$ is a 2-Vector Space regarded as a $\mathcal{VB}$-groupoid over $\{*\} \rightrightarrows \{*\}$, then $\Gamma_{\text{mult}}(\mathcal{V})$ can be identified with V_0, since functors from the trivial groupoid can be identified with the set of units V_0 in V_1.

Example 4.2. In the case of the Tangent Groupoid $T\mathcal{G} \rightrightarrows TM$, a multiplicative section is called a **multiplicative vector field**. The space of multiplicative vector fields is denoted $\mathfrak{X}_{\text{mult}}(\mathcal{G})$.

Example 4.3. In the case of the cotangent groupoid $T^*\mathcal{G} \rightrightarrows A^*$, multiplicative sections are called **multiplicative 1-forms**. The space of multiplicative 1-forms is denoted $\Omega^1_{\text{mult}}(\mathcal{G})$.

Example 4.4. If $\mathcal{G} \ltimes E \rightrightarrows E$ is the $\mathcal{VB}$-groupoid associated to a representation of $\mathcal{G} \rightrightarrows M$ on a vector bundle $E \to M$, then $\Gamma_{\text{mult}}(\mathcal{G} \ltimes E)$ can be identified with with the space of $\mathcal{G}$-invariant sections of E:

$$\Gamma(E)^{\mathcal{G}} = \{e \in \Gamma(E) : g \cdot (e(s(g))) = e(t(g))\}$$

If $(V, e) \in \Gamma_{\text{mult}}(\mathcal{G} \ltimes E)$, since (V, e) is a morphism of Lie groupoids, $\tilde{s} \circ V(g) = e(s(g))$ implies that $V = s^*e$, and from $\tilde{t} \circ V(g) = e(t(g))$ follows that $\tilde{t} \circ V(g) = g \cdot (s^*e)(g) = g \cdot (e(s(g))) = e(t(g))$. Conversely, if $e \in \Gamma(E)^{\mathcal{G}}$, then (s^*e, e) is multiplicative. Indeed, given $(g, h) \in \mathcal{G}^{(2)}$, notice that $h \cdot e(s(h)) = e(t(h)) = e(s(g))$. Thus $(s^*e)(g)(s^*e)(h) = (g, e_{s(g)})(h, e_{s(h)}) = (gh, e_{s(h)}) = (gh, e_{s(gh)}) = (s^*e)(gh)$.

Example 4.5. If $G \ltimes E \rightrightarrows E$ is the action groupoid associated to a G-equivariant vector bundle, then by the preceding example $\Gamma_{\text{mult}}(G \times E) = \Gamma(E)^{G \ltimes M}$. However, since the elements (g, m) act only on elements of E_m, we can identify $\Gamma(E)^{G \ltimes M}$ with $\Gamma(E)^G$.

Proposition 4.1. A $\mathcal{VB}$-isomorphism between $\mathcal{V}_1 \rightrightarrows E_1$ and $\mathcal{V}_2 \rightrightarrows E_2$ over $\mathcal{G} \rightrightarrows M$ induces an isomorphism $\Gamma_{\text{mult}}(\mathcal{V}_1) \cong \Gamma_{\text{mult}}(\mathcal{V}_2)$.

Proof: First, we note that if $(\Psi^{\mathcal{V}}, \Psi^E, \mathrm{id}^{\mathcal{G}}, \mathrm{id}^M)$ is a $\mathcal{VB}$-morphism, and $(V, e) \in \Gamma_{\mathrm{mult}}(\mathcal{V}_1)$, then since $(\Psi^{\mathcal{V}}, Id^{\mathcal{G}})$ and (Ψ^E, Id^M) are vector bundle morphisms, then $(\Psi^{\mathcal{V}} \circ V, \Psi^E \circ e) \in \Gamma(\mathcal{V}_2)$ and it is a functor because it is the composition of two functors. Now, if the $\mathcal{VB}$-morphism is a $\mathcal{VB}$-isomorphism, we can define the inverse map in an analogous manner. The isomorphism thus follows from fiberwise linearity of $(\Psi^{\mathcal{V}}, \mathrm{id}^{\mathcal{G}})$ and (Ψ^E, id^M). □

As a direct consequence of Proposition 4.1. and of the isomorphism between a $\mathcal{VB}$-groupoid $\mathcal{V} \rightrightarrows E$ and the split $\mathcal{VB}$-groupoid $t^*C \oplus s^*E$ determined by a representation up to homotopy, we have an isomorphism $\Gamma_{\mathrm{mult}}(\mathcal{V}) \cong \Gamma_{\mathrm{mult}}(t^*C \oplus s^*E)$ that allows us reduce the study of multiplicative sections to a more computable case.

One consequence of this isomorphism is that we can prove by direct computation that sections of the core determine multiplicative sections of a $\mathcal{VB}$-groupoid in a simple way:

Proposition 4.2. Let $\mathcal{V} \rightrightarrows E$ be a $\mathcal{VB}$-groupoid, with C as its core. If $c \in \Gamma(C)$, then $c^r - c^l \in \Gamma_{\mathrm{mult}}(\mathcal{V})$.

Proof: First, note that given a $\mathcal{VB}$-map $(\Psi^{\mathcal{V}}, \Psi^E, \mathrm{id}^G, \mathrm{id}^M)$ between $\mathcal{VB}$-groupoids $\mathcal{V}_1 \rightrightarrows E_1$ and $\mathcal{V}_2 \rightrightarrows E_2$ preserves left and right-invariant sections of $\mathcal{V}_1 \rightrightarrows E_1$. Indeed, $\Psi^{\mathcal{V}}(c^r(g)) = \Psi^{\mathcal{V}}(c_{t(g)} \cdot \tilde{0}_g) = \Psi^{\mathcal{V}}(c)_{t(g)} \cdot \tilde{0}_g = \Psi^{\mathcal{V}}(c)^r(g)$; the case of a left-invariant section is analogous.

Now, consider a section $c \in \Gamma(C)$. We have

$$
\begin{aligned}
(c^r - c^l)(g) &= (c_{t(g)}, g, 0) - (\Delta^C_g c_{s(g)}, g, -\partial c_{s(g)}) \\
&= (c_{t(g)} - \Delta^C_g c_{s(g)}, g, \partial c_{s(g)})
\end{aligned}
$$

On one hand:

$$
\begin{aligned}
(c^r - c^l)(gh) &= (c_{t(g)} - \Delta^C_{gh} c_{s(h)}, gh, \partial c_{s(h)}) \\
&= (c_{t(g)} - \Delta^C_g \Delta^C_h c_{s(h)} - \Omega_{g,h} \partial c_{s(h)}, gh, \partial c_{s(h)}),
\end{aligned}
$$

where the last equality follows from the second condition of a representation up to homotopy. On the other hand,

$$
\begin{aligned}
(c^r - c^l)(g) \cdot (c^r - c^l)(h) &= (c_{t(g)} - \Delta^C_g c_{s(g)}, g, \partial c_{s(g)}) \cdot (c_{t(h)} - \Delta^C_h c_{s(h)}, h, \partial c_{s(h)}) \\
&= (c_{t(g)} - \Delta^C_g c_{s(g)} + \Delta^C_h c_{t(h)} - \Delta^C_g \Delta^C_h c_{s(h)} - \Omega_{g,h} \partial c_{s(h)}, gh, \partial c_{s(h)}) \\
&= (c_{t(g)} - \Delta^C_g \Delta^C_h c_{s(h)} - \Omega_{g,h} \partial c_{s(h)}, gh, \partial c_{s(h)}),
\end{aligned}
$$

where the last equality follows because of the second condition of a representation up to homotopy, and because since $s(g) = t(h)$, $c_{s(g)} = c_{t(h)}$. Hence $(c^r - c^l)(gh) = (c^r - c^l)(g)(c^r - c^l)(h)$ for any $(g, h) \in \mathcal{G}^{(2)}$. □

The set $\Gamma_{\mathrm{mult}}(\mathcal{V})$ carries more structure than that of a vector space. Since multiplicative sections are functors, we can use natural transformations to turn it into a category.

Definition 4.2. Let $\mathcal{V} \rightrightarrows E$ be a $\mathcal{VB}$-groupoid over $\mathcal{G} \rightrightarrows M$. The **category of multiplicative sections** $\mathrm{Sec}\,(\mathcal{G}, \mathcal{V})$ is the subcategory of $\mathrm{Hom}_{\mathrm{LieGrpd}}\,(\mathcal{G}, \mathcal{V})$ defined as:

- *Objects*: Multiplicative sections (V, e) of $\mathcal{V} \rightrightarrows E$;
- *Morphisms*: Smooth natural transformations $\tau : V \Rightarrow V'$ such that $1_{q_{\mathcal{V}}} \bullet \tau = 1_{id_{\mathcal{G}}}$, where $\bullet$ indicates the horizontal composition of natural transformations. The composition in $\mathrm{Sec}(\mathcal{G}, \mathcal{V})$ is the vertical composition of natural transformations.

Morphisms in $\mathrm{Sec}(\mathcal{G}, \mathcal{V})$ have a simpler description: They are certain sections of the pullback $1^*\mathcal{V}$, which is isomorphic to $C \oplus E$.

Proposition 4.3. Let $\mathcal{V} \rightrightarrows E$ be $\mathcal{VB}$-groupoid, (V, e),$(V', e') \in \mathrm{Sec}\,(\mathcal{G}, \mathcal{V})$. A natural transformation $\tau : V \Rightarrow V'$ is a morphism in $\mathrm{Sec}(\mathcal{G}, \mathcal{V})$ if, and only if $\tau(x) \in \mathcal{V}_{1_x}$, for all $x \in M$.

Proof: Let $1_x \in \mathcal{G}$ be a unit. We have

$$\begin{aligned}(1_{q_\mathcal{V}} \bullet \tau)(x) &= 1_{q_\mathcal{V}}(q_E \circ e'(x))q_\mathcal{V}(\tau(x)) \\ &= (1_{q_\mathcal{V}}(x) \circ q_\mathcal{V}(\tau(x)) \\ &= 1_x \circ q_\mathcal{V}(\tau(x)) \\ &= q_\mathcal{V} \circ \tau_x\end{aligned}$$

whereas $1_{id_\mathcal{G}}(x) = 1_x$. So $1_{q_\mathcal{V}} \bullet \tau = 1_{id_\mathcal{G}} \iff q_\mathcal{V} \circ \tau = 1 : M \to \mathcal{G} \iff \tau(x) \in \mathcal{V}_{1_x}$. □

Conversely, we can describe which sections of $\mathcal{V}$ are natural transformations in $\mathrm{Sec}(\mathcal{G}, \mathcal{V})$. For that, we want to consider our $\mathcal{VB}$-groupoid to be a split one associated to a representation up to homotopy. To do that, first we enhance the isomorphism given in Proposition 4.1.:

Proposition 4.4. Let $\mathcal{V}_1 \rightrightarrows E_1$ and $\mathcal{V}_2 \rightrightarrows E_2$ be isomorphic $\mathcal{VB}$-groupoids over $\mathcal{G} \rightrightarrows M$. Then $\mathrm{Sec}(\mathcal{G}, \mathcal{V}_1) \cong \mathrm{Sec}(\mathcal{G}, \mathcal{V}_2)$.

Proof: Let $(\Psi^\mathcal{V}, \Psi^E, \mathrm{id}^\mathcal{G}, \mathrm{id}^M)$ be a $\mathcal{VB}$-isomorphism, and define the functor $(\Psi^\mathcal{V}, \Psi^E) : \mathrm{Sec}(\mathcal{G}, \mathcal{V}_1) \to \mathrm{Sec}(\mathcal{G}, \mathcal{V}_2)$ by the following: on objects, it maps a multiplicative section (V, e) and maps into $(\Psi^\mathcal{V} \circ V, \Psi^E \circ e)$, and on morphisms it takes a natural transformation $\tau : V \Rightarrow V'$ and maps to $\Psi^\mathcal{V}(\tau_x) : \Psi^\mathcal{V} \circ V \Rightarrow \Psi^\mathcal{V} \circ V'$.

We already know by Proposition 4.1. that this defines a bijection on the sets of objects. To see that $\Psi^\mathcal{V}(\tau_x)$ is a well-defined natural transformation, given $g : x \to y$ in $\mathcal{G}$, it follows from functoriality of $\Psi^\mathcal{V}$ that $V(g) \circ \tau_x = \tau_y \circ V'(g)$ implies

$$\begin{aligned}\Psi^\mathcal{V}(V(g)) \circ \Psi^\mathcal{V}(\tau_x) &= \Psi^\mathcal{V}(V(g) \circ \tau_x) \\ &= \Psi^\mathcal{V}(\tau_y \circ V'(g)) \\ &= \Psi^\mathcal{V}(\tau_y) \circ \Psi^\mathcal{V}(V'(g)).\end{aligned}$$

Now, by Proposition 4.3., we need to prove that $\Psi^\mathcal{V}(\tau(x)) \in \mathcal{V}_{2_{1_x}}$, but this is a consequence of the fact that $\Psi^\mathcal{V}$ is a vector bundle morphism, and $\tau_x \in \mathcal{V}_{1_x}$. It is a functor since functoriality of $\Psi^\mathcal{V}$ implies $\Psi^\mathcal{V}(\sigma \circ \tau)(x) = \Psi^\mathcal{V}(\tau_x) \circ \Psi^\mathcal{V}(\sigma_x)$ and $\Psi^\mathcal{V}(1_x) = 1_x$ for σ_x, τ_x, 1_x natural transformations on $\mathrm{Sec}(\mathcal{G}, \mathcal{V}_1)$. □

Proposition 4.5. Let (V, e), $(V', e') \in \Gamma_{\mathrm{mult}}(\mathcal{V})$, and suppose that $\tau : M \to \mathcal{V}$ is a smooth map with $q_\mathcal{V} \circ \tau = 1 : M \to \mathcal{G}$. Identify τ with a section $(c_0, e_0) \in \Gamma(1^*\mathcal{V}) \cong \Gamma(C) \oplus \Gamma(E)$. Then τ defines a morphism $\tau : V \Rightarrow V'$ if, and only if the following conditions hold:

1. $e_0 = e$
2. $e'_0 = e_0 + \partial(c_0)$
3. $V' = V + c_0^r - c_0^l$

Proof: Let $\mathcal{V} = t^*C \oplus s^*E \rightrightarrows E$ be the $\mathcal{VB}$-groupoid associated with a representation up to homotopy. A smooth map $\tau : M \to \mathcal{V}$ is a natural transformation $V \Rightarrow V'$ if, and only if the following hold:

(i) $\tilde{s} \circ \tau = e$

(ii) $\tilde{t} \circ \tau = e'$

(iii) $\tau(g)$ is natural in g with respect to (V, e) and (V', e').

For $x \in M$, we have:

$$\tilde{s} \circ \tau(x) = \tilde{s}(c_0(x), 1_x, e_0(x)) = e_0(x)$$

$$\tilde{t} \circ \tau(x) = \tilde{t}(c_0, 1_x, e_0(x)) = \partial(c_0(x)) + e_0(x)$$

Hence $i. \Leftrightarrow 1.$ and $ii. \Leftrightarrow 2.$; Now, assume that $\tau : M \to V$ satisfies both $i.$ and $ii.$ Naturality of $\tau : V \Rightarrow V'$ means that for $g \in \mathcal{G}$, the following diagram commutes:

$$\begin{array}{ccc} e(s(g)) & \xrightarrow{\tau_{s(g)}} & e'(s(g)) \\ \Big\downarrow{\scriptstyle V(g)} & & \Big\downarrow{\scriptstyle V(g')} \\ e(t(g)) & \xrightarrow{\tau_{t(g)}} & e'(t(g)). \end{array}$$

Since $\mathcal{V} = t^*C \oplus s^*E$, we write $V(g) = (c_{t(g)}, g, e_{s(g)})$ and $V'(g) = (c'_{t(g)}, g, e'_{s(g)})$.
On one hand:

$$\begin{aligned} \tau(t(g))V(g) &= (c_0(t(g)), 1_{t(g)}, e_0(t(g))) \cdot (c_{t(g)}, g, e_{s(g)}) \\ &= (c_0(t(g)) + \Delta^C_{1_{t(g)}}(c_{t(g)}), g, e_{s(g)}) \\ &= (c_0(t(g)), g, 0) + (c_{t(g)}, g, e_{s(g)}) \\ &= c^r(g) + V(g). \end{aligned}$$

On the other hand:

$$\begin{aligned} V'(g) \cdot \tau(s(g)) &= (c'_{t(g)}, g, e'_{t(g)}) \cdot (c_0(s(g)), 1_{s(g)}, e_0(s(g))) \\ &= (c'_{t(g)} + \Delta^C_g(c_0(s(g))), g, e_0(s(g))) \\ &= (c'_{t(g)} + \Delta^C_g(c_0(s(g))), g, e'_{s(g)} - \partial c_0(s(g))) \\ &= (c'_{t(g)}, g, e'_{s(g)}) + (\Delta^C_g(c_0(s(g))), g, -\partial c_0(s(g))) \\ &= V'(g) + c^l(g). \end{aligned}$$

Thus $V' = V + c^r_0 - c^l_0$.

Definition 4.3. Let $\mathcal{V} \rightrightarrows E$ be a Lie groupoid over $\mathcal{G} \rightrightarrows M$ with core C. The **complex of multiplicative sections** of $\mathcal{V}$, denoted $C^\bullet_{\text{mult}}(\mathcal{V})$ is the 2-Term complex of vector spaces

$$\begin{aligned} \delta : \Gamma(C) &\to \Gamma_{\text{mult}}(\mathcal{V}) \\ c &\mapsto c^r - c^l. \end{aligned}$$

Theorem 4.1. The category $\text{Sec}(\mathcal{G}, \mathcal{V})$ is isomorphic to the underlying category of the 2-Vector space $\Gamma(C) \oplus \Gamma_{\text{mult}}(\mathcal{V}) \rightrightarrows \Gamma_{\text{mult}}(\mathcal{V})$.

Proof: We define the following functor candidates: one, from $\text{Sec}(\mathcal{G}, \mathcal{V})$ to $\Gamma(C) \oplus \Gamma_{\text{mult}}(\mathcal{V}) \rightrightarrows \Gamma_{\text{mult}}(\mathcal{V})$, that is the identity on objects and mapping a natural transformation $\tau : V \Rightarrow V'$ to an arrow $(c_0, (V, e_0))$, where τ is identified with (c_0, e_0). The inverse, from $\Gamma(C) \oplus \Gamma_{\text{mult}}(\mathcal{V}) \rightrightarrows \Gamma_{\text{mult}}(\mathcal{V})$ to $\text{Sec}(\mathcal{G}, \mathcal{V})$, that is also the identity on objects, and that takes an arrow $(c, (V, e))$ to the natural transformation given by $(c, e) : V \Rightarrow V + c^r - c^l$. By Propositions 4.3. and 4.5., both are bijections on objects and morphisms.

We prove now that they are functors. Let (V, e), (V', e'), (V'', e'') be multiplicative sections, and $\tau : V \Rightarrow V'$, $\sigma : V' \Rightarrow V''$ be composable morphisms in $\text{Sec}(\mathcal{G}, \mathcal{V})$ with $\tau = (c_0, e)$, $\sigma = (c_1, e')$ in $\Gamma(C) \oplus \Gamma(E)$. Vertical composition of natural transformations gives the following diagram for $\sigma \circ \tau$:

$$\begin{array}{ccc} e_{s(g)} & \xrightarrow{V(g)} & e_{t(g)} \\ {\scriptstyle \sigma(s(g))\tau(s(g))}\Big\downarrow & & \Big\downarrow{\scriptstyle \sigma(t(g))\tau(t(g))} \\ e''_{s(g)} & \xrightarrow{V''(g)} & e''_{t(g)}. \end{array}$$

To prove that the composition corresponding to $\sigma \circ \tau$ is the composition $(c_1, (V', e'))(c_0, (V, e)) := (c_1 + c_0, (V, e))$, first note that by Proposition 3.4, $\tilde{s}(c_1, (V', e')) = V' = V + \delta(c_0) = \tilde{t}(c_0, (V, e))$. So, by the same proposition, we are left with proving that

$$\sigma(t(g))\tau(t(g))V(g) = (c_1 + c_0)^r(g) + V(g)$$
$$V''(G)\sigma(s(g))\tau(s(g)) = V''(g) + (c_1 + c_0)^l(g),$$

For the first identity, we have by the proof of Proposition 3.4. that

$$\tau(t(g))V(g) = c_0^r(g) + V(g) = (c_0(t(g)) + c(t(g)), g, e(s(g)))$$

consequently

$$\begin{aligned}\sigma(t(g))\tau(t(g))V(g) &= (c_1(t(g)), 1_{t(g)}, e'(t(g)))(c_0(t(g)) + c(t(g)), g, e(s(g)))\\ &= (c_1(t(g)) + \Delta^C_{1_{t(g)}}(c_0(t(g)) + c(t(g)), g, e(s(g)))\\ &= (c_1(t(g)) + c_0(t(g)), g, 0) + (c(t(g)), g, e(s(g)))\\ &= (c_1 + c_0)^r(g) + V(g).\end{aligned}$$

We proceed analogously for the second identity. Note that:

$$\begin{aligned}V''(g)\sigma(s(g)) = V''(g) + c_1^l(g) &= (\bar{c}(t(g)), g, \bar{e}(s(g))) + (\Delta^C_g(\bar{c}_{s(g)}), g, -\partial(\bar{c}_{s(g)})\\ &= (\bar{c}(t(g)) + \Delta^C_g(\bar{c}_{s(g)}), g, \bar{e}(s(g)) - \partial(\bar{c}_{s(g)})),\end{aligned}$$

consequently

$$\begin{aligned}V''(g)\sigma(s(g))\tau(s(g)) &= (\bar{c}(t(g)) + \Delta^C_g(\bar{c}_{s(g)})), g, \bar{e}(s(g)) - \partial(\bar{c}_{s(g)}))(c_0(s(g)), 1_{s(g)}, e(s(g)))\\ &= (\bar{c}(t(g)) + \Delta^C_g(c_1(s(g)) + c_0(s(g))), g, e_0(s(g)))\\ &= (\bar{c}(t(g)) + \Delta^C_g(c_1(s(g)) + c_0(s(g))), g, e''(s(g)) - \partial(c_0 + c_1))\\ &= (\bar{c}(t(g)), g, e''(s(g))) + (\Delta^C_g(c_1(s(g)) + c_0(s(g))), g, \partial(c_0 + c_1))\\ &= V''(g) + (c_1 + c_0)^l(g),\end{aligned}$$

where the change in the third coordinate from the second to the third line follows from $e' = e + \partial(c_0)$ and $e'' = e' + \partial(c_1)$. □

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