Geometric Quantization

Stiven Ramírez Quintero

December of 2022

1 Introduction

I'm going to describe the process called Geometric Quantization which is based in using tools of sympletic geometric to associate a quantum system to a classical system (such as Harmonic Oscillator).The process consists of three different steps, first we consider the configuration space Q of the classical system and then the cotangent space $M = T^*Q$ which has a natural structure of symplectic manifold, i.e, it has a natural closed 2 form expressible locally like $\omega|_U = \sum_a dq^a \wedge dp_a$ where $(q^a, p_a) \in U \subset M = T^*Q$ (This is thanks to Darboux's theorem). Note that $q^a \in Q$ are the positions components of the systems and p_a are the momentum for every position component.

We use prequantization to obtain a Hilbert space that result not be completely satisfactory because of correspondent Hilbert space is too big and many of the observables aren't correctly reproduced. That's way, we correct this by a process called polarization that reduces of number of variables of operator to finally do the quantization well.

2 Observations

(1) It's important to keep in mind that not every manifold can be quantizable and not every symplectic manifold as well can be quantizable, in order for a symplectic manifold be quantizable is necessary that the manifold satisfies the condition of integrality discuss in the next sections.

(2) Not every symplectic manifold have a form of a cotangent space (in some way, it looks like a cotangent space but locally in a finite number of cards, thanks to Darboux's Theorem).

(3) Every classical system can be quantizable but if we have a quantum system it's not the case that it came from a classical system, one classical example is the spin.

3 Why is the geometric quantization important-useful?

GC is important because it enables us to give another interpretation to classical mechanic, i.e, we now can understand the behavior of a Hamiltonian systems which is a mathematical description of a big object from of point of view of the physics used to understand micro particles.

4 What is a classical system?

A classical system is a mathematical descriptions of a macro physical phenomenon, in this work we will only focusing our attention to conservative Hamiltonian Systems which makes part of classical systems

Examples of Hamiltonian systems

We will only give an example of a Hamiltonian system called Harmonic oscillator but if you are interested in knowing more of these examples, see P for more like Kepler problem and Particle in a field.

In the case of the Harmonic Oscillator we have

$$M = T^*Q = T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n}$$

then we have that we configuration space is $Q = \mathbb{R}^n$ and the phase space is $M = \mathbb{R}^{2n}$.

So, this manifold is also equipped with the 2-form $\omega = dq^a \wedge dp_a$. Also we have as a a hamiltonian function $H \in C^\infty(M \cong \mathbb{R}^{2n})$ callled also observable given by

$$H(q,p) = \frac{1}{2}(||q||^2 + ||p||^2)$$

.

And a vector field associate with H which we will called Hamiltonian vector field:

$$X_H = p_a \frac{\partial}{\partial q^a} - q^a \frac{\partial}{\partial p_a}$$

Observation

Following the same idea, for every observable $f \in C^\infty(M)$ we have a field vector X_f called Hamiltonian vector field. This correspondence turns out to be an homomorphism and we can construct and isomorphism making some adjust and using the machinery of the symplectic manifold which has special characteristics (this is important later in the text to construct the precuantization function).

5 Geometric interpretation of a classical system

Given a (conservative) Hamiltonian System, we can associate a geometric object to that of the following way. Let Q the configurations space of the Hamiltonian System which is a manifold. We then consider the cotangent space of the configuration space, which is $M = T^*Q$ and it's called the phase space.

Functions on $M = T^*Q$ are called observables, i.e $f(q,p) \in C^\infty(T^*Q)$. Since the Hamiltonian system comes with a Hamiltonian function H, this function makes part of the observables, here $H(q,p) \in C^\infty(M = T^*Q)$ where $(q,p) \in T^*Q$. And the manifold obtained from a Hamiltonian system is usually refer like (M, ω, H)

Observation I'm doing slight notation abuses calling elements in T^*Q like (q, p) because elements in T^*Q are expressible like $p_a dq^a$, but we will usually use the above notation.

Like we saw early, the cotangent space has a natural non degenerate, closed 2-form which is given by $\omega = \sum_i dq^i \wedge dp_i$ where $(q^1, ..., q^n)$ are the positions coordinates of the movement described by the Hamiltonian system and $(p_1, ..., p_n)$ are the coordinates of the momentum associate to each of the component of the coordinate of position. (this notations of super indices are taken by convention and it-s called the Einstein notation).

Given that the symplectic manifold has a 2-form, we can consider the vector field X_f for a observable $f \in C^\infty(M)$ as the 1-form:
$df = -\omega(X_f, \cdot)$

Locally, the vector field looks like

$$X_f|_U = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$$

Notation The vector field associate with a observable is called a Hamiltonian vector field.

Another thing we can say is the Hamiltonian vector fields define an infinity dimensional algebra if we define

$$\{X_f, X_g\} = \omega(X_f, X_g) = -df(X_g)$$

for f, g observables.

6 Review of symplectic geometry

In this section I'm going to describe what I will need of Symplectic Geometric in order to describe of process of Quantization. Clearly it's a uncompleted description of Symplectic Geometric because all will be focus only to describe GC. Symplectic geometric is a full interesting subject by its own, and for more detailed description of this see [7].

Differential 2-form

Let M a differentiable manifold. A differential 2-form is a function $\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathbb{R}$ that is bi-linear and antisimetric. Note that $\omega \in (\mathfrak{X}^2(M))^*$

Non-degenerate 2-form

A non-degenerate 2-form is a 2-form such that if $\omega(X, Y) = 0$ for every $Y \in \mathfrak{X}(M)$ that means that $X = 0$.

Closed 2-form

A closed 2-form is a 2-form ω that satisfies $d\omega = 0$ where d is the exterior derivative:

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

Observation

If a 2-form is closed, by Poincare's lemma, we have that locally there exists a symplectic potential, i.e. $\omega|_U = d\theta$ for every $U \subset M$.

Symplectic manifold

A symplectic manifold is a differentiable manifold M equipped with a non-degenerate and closed 2-form ω. We denote this manifold by (M, ω) .

Examples of simplectic manifolds

2n dimensional euclidean space ($M = T^*(\mathbb{R}^n) = \mathbb{R}^{2n}$). This space come with a natural 2-form ω such that if $(q, p) \in T^*\mathbb{R}^n$ is such that $q = (q_1, q_2, ..., q_n), p = (p_1, p_2, ..., p_n) \in \mathbb{R}^n$ then the sympectic form is given by

$$\omega = \sum_i dq_i \wedge dp_i$$

In this case is easy to verify the conditions of the symplectic manifold because the wedge is billinear and antisymetric as well.

Cotangent space. A very important example of a simplectic manifold is the cotangente space of a manifold Q where Q is any manifold. This is particular important in this work because we are quantizating the cotangent space of a configuration space Q that is associate with a Hamiltonian system. The 2-form is given similarity from the before example, we are considering essentially the same 2-form.

2-Sphere ($\mathbb{S}^2$). If we consider $\mathbb{S}^2$ like the set of vector in $\mathbb{R}^3$ with norm equal to one, we can define a bilinear, closed 2-form like

$$\omega_p : T_p\mathbb{S}^2 \times T_p\mathbb{S}^2 \to \mathbb{R}$$

given by $\omega_p(u, v) = w_p(p, u \times v)$

Interesting fact about symplectic manifolds

(1) Every symplectic manifold has even dimension. This is consequence of the antisymmetry of the 2-form:

Let's suppose that $\omega : V \times V \to \mathbb{R}$ is a 2-form where V is a finite vector space with base $B = \{v_1, ..., v_n\}$ and let $(a_{ij}) = \langle v_j, v_i\rangle$ be a matrix representing ω in the base B, then as ω is anti-symmetric, we need to have that $[\omega]_B = -[\omega]_B^t$ so $\det[\omega]_B = det(-[\omega]_B^t) = (-1)^n \det[\omega]_B$ and $n = 2k$ for some $k \in \mathbb{N}$

Important observation

In order to define Geometric Quantization correctly, we need to use a connection of the complex line bundle. So I will need to introduce the concept of the complex line bundle and then the connection associate to this bundle, which I am going to do now.

Complex line bundle

A complex line bundle over a manifold M is another manifold L together with a surjective function $\pi : L \to M$ that satisfies the following conditions:

(i) For every $a \in M$ we have that $\pi^{-1}(a)$ is a complex vector space of dimension one. (i.e. $\pi^{-1}(a) \cong \mathbb{C}$ for every $a \in M$ and so the fibers are lines)

(ii) π is locally trivial, that means that there exist a open covering $\{U_i\}$ of M that such that the functions $\tau_i : U_i \times \mathbb{C} \to \pi^{-1}(U_i)$ given by $(m, z) \mapsto z s_i(m)$ are diffeomorphism, where $s_i : U_i \to \pi^{-1}(U_i)$ are called unitary sections (we will define sections below).

Section of the complex line bundle

Sections of the complex line bundle are function $s : M \to L$ such that $\pi \circ s = Id$. In the previous definition s_i are section and then they are functions such the $\pi \circ s_i = Id$.

The set of every sections of a manifold M associate with a complex line bundle will be denoted by $\Gamma(M)$ and it turns out that $\Gamma(M)$ is an algebra.

Complex lines bundle doesn't always exist for a given symplectic manifold, we can see in "BAYKARA" and "CAROSO" that one necessary condition to this particular vector bundle exist is that $\left[\frac{\omega}{2\pi}\right] \in H_{dR}(M, \mathbb{Z})$ which is equivalent to Weill's integrability condition, that says that the complex line bundle for a symplectic manifold there exist if

$$\int_{\Sigma} \omega \in 2\pi\mathbb{Z}$$

for every closed surface $\Sigma \subset M$.

Connection

For a general vector bundle we can define a connection, in this particular case that we have a complex line bundle, we are going to introduce a connection of this vector bundle which will be key for the correct definition of the prequantization function that satisfy the Dirac's conditions (we will see this condition in some moments).

A connection is a way to derivative field vector with respect to field vector and is defined by a $\mathbb{R}-$ bi-linear function $\bigtriangledown : \mathfrak{X}(M) \times \Gamma(M) \to \Gamma(M)$ given by $(X, e) \to \bigtriangledown_X e$ that satisfies:

(i) $\bigtriangledown_{fX} e = f \bigtriangledown_X e$

(ii) $\bigtriangledown_X (f \cdot e) = f \bigtriangledown_X e + X(f) \cdot e$

7 Prequantization

The idea of Prequantization is to associate a Hilbert space to one manifold, that Hilbert space will have operators self ad-joints acting on the Hilbert space. The idea is to pick one observable in the manifold and then to find an operator self ad-joint in the correspondent Hilbert space, to do this we need to consider the complex line bundle of the manifold and the Hilbert space will be the sections of this complex line bundle.

We want to construct a function $f \mapsto \widehat{f}$ from classical observables to operators in a Hilbert space ($C^\infty(M) \to \mathbb{H}$) that has a good behavior, that is, being linear and respecting of Poisson bracket what was defined before, i.e, our function needs to satisfy (Dirac's conditions):

(1) $f \mapsto \widehat{f}$ is $\mathbb{R}-$linear.
(2) If f is a constant observable, then $\widehat{f} = cI$ (where I is the identity operator in the Hilbert space)
(3) If $\{f, g\} = h$, then $[\widehat{f}, \widehat{f}] = -i\widehat{h}$ ($-i$ appears here by convention)

There are several attempts to defined that function and we are going to explore some of them until arrive to the correct one.

First attempt to define $f \mapsto \widehat{f}$:

The first natural way of defining $f \mapsto \widehat{f}$ is by doing $\widehat{f} = -iX_f$ and we can notice that this definition satisfies (2) but we can notice that it doesn't satisfy one of the condition we want to, the condition (1), because

$\widehat{c} = -iX_c = 0 \neq cI$.

Second attempt to define $f \mapsto \widehat{f}$:

Another way we can think works is by defining $\widehat{f} = -iX_f + f$. This satisfy (1) condition but doesn't satisfy (2).

Third attempt to define $f \mapsto \widehat{f}$:

To do our third attempt we are going to suppose that ω has a globally symplectic potential, i.e, there exists θ such that $\omega = d\theta$. In this case then, we define $\widehat{f} = -iX_f + f - \theta(f)$.

We can check that this definition of $\widehat{f}$ satisfies the properties that we want, but the problem is the choice and the global existence of the symplectic potential θ. To solve this, we need to introduce a connection of the complex line bundle that was introduced briefly.

Correct way to define $f \mapsto \widehat{f}$:

Let's suppose that there exist a complex line bundle $L \to M$ and a connection $\bigtriangledown : \mathfrak{X}(M) \times \Gamma(M) \to \Gamma(M)$ then it turns out that if we define $\widehat{f} : \Gamma(M) \to \Gamma(M)$ like $\widehat{f} = -i \bigtriangledown_{X_f} + f$ for every observable $f \in C^{\infty}(M)$, this definition satisfy the Dirac's conditions and it doesn't have the problems of the potential symplectic as before, then this is the correct way to find a Hilbert space (the space of the quadratic integrable sections with the Liouville measure of the symplectic manifold).

This space will turn out to be not good enough and it will be necessary to do some adjustment that we will see in the next sections. That's way what we did here it's called prequntization because the Hilbert space obtained is still incorrect, why? this Hilbert space consist of quadratic integral function with respect to the Liouville measure of the symplectic manifold, that are observables and then they depend of position and momentum, so we only need function that depend of one of the variables but not both.

8 Two examples of prequantization

(1) If we take $Q = \mathbb{R}^n$ the configuration space and $M = T^*Q \cong \mathbb{R}^{2n}$ the phase space, we have then that $\omega = dq^i \wedge dp_i$ and $\theta = p_i dq^i$, what means $\omega = d\theta$ and ω is globally exact so in this case we can define $\widehat{f} = -iX_f + \theta(X_f) + f$. (in this case we are taking the trivial complex line bundle $L = M \times \mathbb{C}$)

In this case prequantization is possible because (M, ω) trivially satisfies the condition of integrability because if S is a close surface of M then $\partial S = 0$ an so

$$\int_{\partial S} w = 0 = 2\pi \cdot 0$$

Observation The above example applies for every symplectic manifold with a global symplectic potential (in particular cotangent spaces), in this case, we can conclude then that every symplectic manifold with a global symplectic potential can be prequantizable and the way to to this is similar to the above description.

(2) Now, consider $M = \mathbb{S}^2$, it's clear that $\mathbb{S}^2$ is not a cotangent space of a configuration space, so we can't use the before description. But in this case the 2-sphere came equipped with a natural 2-form that is a volume and we have

$$\omega = s^s \sin(\theta) d\theta \wedge d\phi$$

.

Where s is the radio of the 2-sphere in sphere coordinates (θ, ϕ)

So we can corroborate that the integrability condition (Weill's condition) is satisfied and we can consider our Hilbert space as the quadratic-integrable functions in $\mathbb{S}^2$, i.e. $H(\mathbb{S}^2)$ and in this case our complex line bundle can be taken of the following way:

We look at $\mathbb{S}^2$ like $\mathbb{CP}1$, the space of complex lines across the origin of $\mathbb{C}^2$ and one consider L the space obtained by taking one line for every point in $\mathbb{CP}1$, i.e, for each represent in $\mathbb{CP}1$ we take a line and this is the space L

9 Problems with Prequantization

1) We can't always find a global symplectic potential θ for every symplectic manifold ($\omega = d\theta$). which is used to construct the precuantization function $f \mapsto \widehat{f}$ describe above. For example, if the manifold is compact, such θ doesn't exist. (if the symplectic manifold has the form of a cotangent space then always exist a global symplectic potential)

2) Our definition of the function $f \mapsto \widehat{f}$ involve a choice of the symplectic potential (the symplectic potential isn't unique because if θ is a symplectic potential then $\theta + du$ is also potential since $ddu = 0$). Then we need to ensure that our definitions doesn't depend of the choice of the symplectic potential.

3) The precuantization process generate a Hilbert space as it was showed in the previous section but it turn out that the Hilbert space is not good enough because is too big in the sense that has ad-joint operator without much significant physic

4) We have dependence in position and momentum and we only want dependence in position or momentum but not both because Hilbert space consist of quantum operators of n variables (it will be achieve with polarization)

10 Solving the problems of prequantization: Polarization

As we saw before, prequantization has problem that has to be solve in order to have a correct quantization, to do that we need to introduce a polarization because it will allows us to cut down the amount of variables of the operator, so far, the operator constructed in the precuantization process were operators in 2n variables but in practice, since the Hilbert space is the Hilbert space of a quantum system, then the operator only need to depend on n variables, it is described by another condition of the function $f \mapsto \hat{f}$, which is:

If $\{f_i\}$ is a complete set of observables then $\{\hat{f}\}$ is a complete set of wave functions (operators in the Hilbert space)

The before conditions restring of amount of variables that the wave functions need to have and it makes necessary then to introduce the concept of polarization and it will be that possible with a further correction.

Definition A polarization of a symplectic manifold (M, ω) of dimension $2n$ is a $n-$dimensional sub-bundle P of the complex tangent space $TM^{\mathbb{C}}$, (here $TM^{\mathbb{C}}$ is the complexification of TM: $TM \otimes \mathbb{C} \cong TM \oplus iTM$) i.e $P \subset TM^{\mathbb{C}}$ such that

(1) if $X, Y \in P$ then $[X, Y] \in P$ (P is a involutive distribution)
(2) $\omega(X, Y) = 0$ for every $X, Y \in P$ (P is a lagranian distribution or isotropic).
(3) $D_a = P_a \cap \bar{P}_a \cap TM$ has a constant rank.

Observation One can consider real polarization but real polarization doesn't always exist, for instance, there isn't a real polarization for the 2-sphere since ....

Definition A polarization is called a Kalher polarization if $P_a \cap \bar{P}_a = \{0\}$

Definition A polarization is called a integral polarization if for every $a \in M$ there exists a neighborhood $U \subset M$ and a sub-manifold $N \subset M$ such that $P_a = T_a M$

Definition The set of maximal sub-manifolds $N \subset M$ such that $P_a = T_a N$ is denoted by M/D

Observation Note that M/D is not always a manifold

Definition If P is a smooth manifold and P is a smooth function then a polarization P is called reducible.

Definition A polarized section (function) with respect to the polarization P is a section $s \in \Gamma(L)$ (observable f) such that $\nabla_X s = 0$ ($L_X(f) = 0$) for every $X \in P$. Intuitively a polarized section is a section what is constant in the fibers of the polarization. The set of polarized function form a ring

Observation Polarized sections not always exists, here Im not entering in detail of this fact because Iam going to consider polarization where this polarized section always exist as the case of kalher polarization and the vertical polarization of $M = T^*\mathbb{R}^n$

That we have constructed so far was a prehilbert space of the square integrable sections over the complex line bundle $\Gamma(L)$ and it turned out to be too big.

Now we can consider several attempts to the correct Hilbert space, one of them is the set of polarized, quadratic-integrable sections

$$\{s \in \Gamma(L) : \nabla_X s = 0, \forall X \in P\}$$

Which result to be a R module where P is the ring of polarized functions. But in this case we need to define how to calculate integral over the polarized section so, we need a form of volume. In the prequantum space we had P but it doesn't work in this case. There are recipes for every case, I mean, different techniques to take the form volume and I will explore in detail the form volume in the case where we have a kalher polarization since in this case it will be more apparent.

To construct the Hilbert space is not enough, some adjust has to be done, I will quantize only symplectic manifold where we don't have so much problems. Some of the problems we need to solve are:

(1) Nothing guarantees that if s is polarized then $\nabla_X s$ is polarised where $X \in P$.

(2) Since we are dealing with a Hilbert space is then necessary to define a inner product of two polarized section, so we can properly talk about square integrable sections.

(3) To define inner product we need to integrate, but it will not possible to integrate over the symplectic manifold M (over the natural form of volume given by the symplectic manifold), we need to find a correct manifold to integrate over.

11 Example of Quantization

Bargmann-Fock Representation Let's suppose we have a kalher polarization over a sympletic manifold M with symplectic form ω (This can be the phase space of a configuration space, to be do it easier I'm going to do that in this case) then, we know that $\omega = dq^i \wedge dp_i$ and we can consider

$$P = \{\frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, ..., \frac{\partial}{\partial \bar{z}_n}\}$$

where $z_j = p_j + iq^j$. Consider the section s_e given by

$$s_e(a) = (a, \exp(-\sum_{j=1}^{n} \frac{\pi}{2}\bar{z}_j z_j)) = \exp(\frac{-\pi}{2}||z||^2)s_1(a)$$

where $s_1(a) = (a, 1), a \in M$

In this case

$$\nabla_X s_e = 2\pi i \alpha(X) s_e$$

where $X = X^j \frac{\partial}{\partial \bar{z}_j}$ and $\alpha = \frac{i}{2}\sum_j \bar{z}_j dz_j$

and every section s has the form $s = f s_e$ where $f \in C^\infty(M)$. After analysis, we can notice that the Hilbert space is

$$\mathbb{H}_p = \{f \in \mathbb{O}(\mathbb{C})^n | \int_{\mathbb{C}^n} |f|^2 \exp(-\pi||z||^2) dvol < \infty\}$$

with inner product

$$\langle f, g\rangle = \int_{\mathbb{C}^n} \bar{f} g \exp(-\pi||z||^2) dvol$$

for $f, g \in \mathbb{H}_p$

and it turns out to be the correct representation of the space $M = T^*\mathbb{R}^n$.

References

[1] N. Woodhouse "Geometric Quantization". Clarendon press. 1991.

[2] Andrea Carosso. "Geometric Quantization".

[3] Zihni Kaan Baykara. "Geometry quantization".

[4] Martin Schottenloher. "Lecture Notes on Geometric Quantization"

[5] Joseph Geraci. "An Introduction To Geometric Prequantization".

[6] Matthias Blau. "Symplectic Geometry and Geometric Quantization".

[7] Ana Cannas da Silva. "Lectures on Symplectic Geometry".