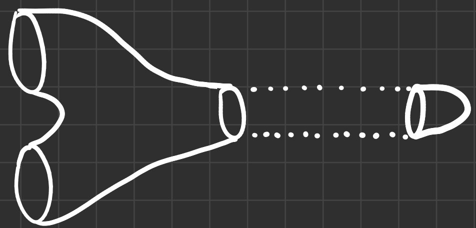
Cobordism Groups

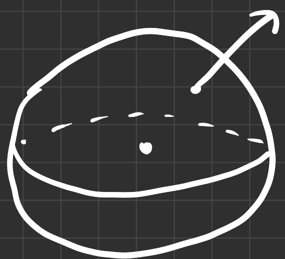
&

The Pontryagin Construction



- $\mathbb{S}^k = \mathbb{R}^k \cup \{\infty\}$
- we want to understand closed $M^n \subseteq \mathbb{R}^{n+k}$

def. A **(normal) framing** of $M^n \subseteq \mathbb{R}^{n+k}$ is a collection $X^1, \dots, X^k \in \Gamma(TM^\perp)$ such that $X^1_p, \dots, X^k_p \in T_pM^\perp$ forms a basis for every $p \in M$. A **framed submanifold** is a submanifold with a fixed framing.

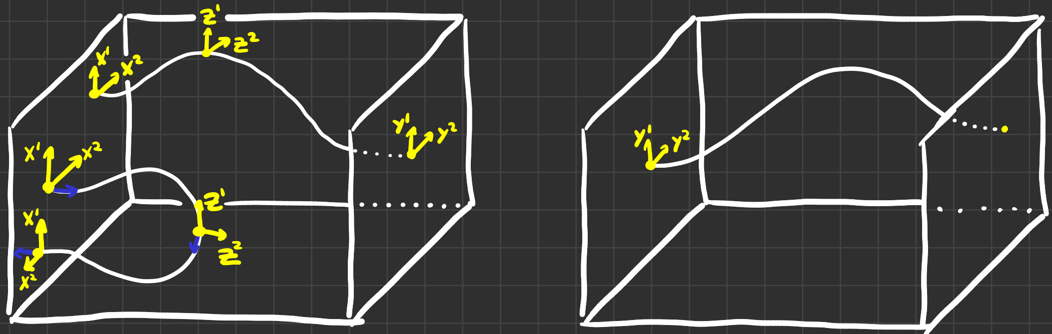


$\hookrightarrow$ no framing!

def. A **(framed) cobordism** $W: M^n \Rightarrow N^n$ is a framed submanifold $W \subseteq \mathbb{R}^{n+k} \times I$ with framing $z^1, \ldots, z^k \in \Gamma(TW^\perp)$ such that

$$\partial W = M \times \{0\} \amalg N \times \{1\}$$

$$z^i_{(p,0)} = x^i_p \qquad z^i_{(p,1)} = y^i_p$$



- $q \in \mathbb{S}^k$ regular value of $f: \mathbb{S}^{n+k} \to \mathbb{S}^k$
- $T_p f^{-1}(q) = \ker(df_p : T_p \mathbb{S}^{n+k} \to T_q \mathbb{S}^k)$

$$df_p : T_p f^{-1}(q)^{\perp} \xrightarrow{\sim} T_q \mathbb{S}^k$$

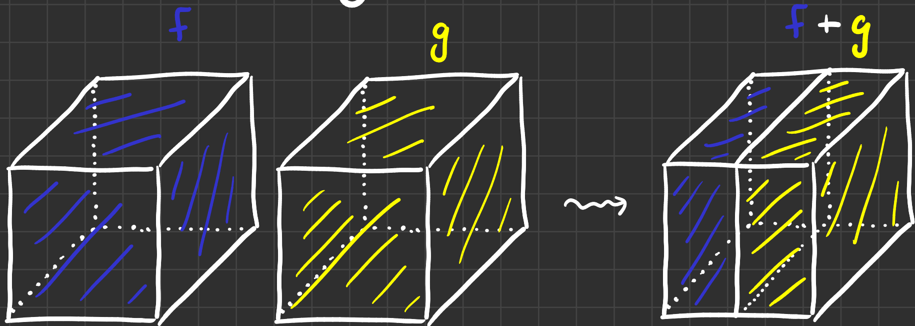
- any <u>positively</u> <u>oriented</u> basis $v_1, \dots, v_k \in T_q \mathbb{S}^k$ induces

$$X_p^i = (df_p)^{-1} v_i$$

prop (Pontryagin, 55). The cobordism class of $f^{-1}(q)$ is independant of the choice of q and $v_1, \dots, v_k$. If $f \sim g$ then $f^{-1}(q)$ and $g^{-1}(q')$ are cobordant!

$$\begin{array}{ccc} \pi_{n+k}(S^k) & \xrightarrow{\;pt\;} & \Omega_n^{Fr(k)} \\ \| & & \| \\ [S^{n+k}, S^k] & & \{M^n \subseteq \mathbb{R}^{n+k} \text{ closed}\}/\simeq \end{array}$$

- $\pi_{n+k}(S^k)$ is a group!



- $\Omega_n^{Fr(K)}$ is also an Abelian group!

flip the sign of the first basis vector

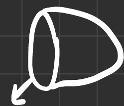
$$[M] + [N] = [M \sqcup N] \qquad 0 = [\emptyset] \qquad -[M] = [\overline{M}]$$

teo (Pontryagin, 55). The map $pt: \pi_{n+k}(S^k) \longrightarrow \Omega_n^{Fr(k)}$ is a group isomorphism!

$$\Omega_0^{Fr(k)} \cong \mathbb{Z}$$

$$\underbrace{\overset{+}{\bullet}\ \overset{+}{\bullet} \cdots \overset{+}{\bullet}}_{n \text{ points}} \longleftrightarrow n$$

$$\Omega_1^{Fr(1)} = 0$$



	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3 \times \mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

- $f^{-1}(q) \subseteq \mathbb{R}^{n+k}$, $f : S^{n+k} \longrightarrow S^k$

independence of $v_1, \dots, v_k \leftrightsquigarrow$ connexivity of $GL^+_k(\mathbb{R})$

- $h : f \Rightarrow g$, $q \in S^k$ regular value of f and g
- if q is a regular value of h take

$$h^{-1}(q) : f^{-1}(q) \Rightarrow g^{-1}(q)$$

- in general we can always find $q' \in S^k$ regular value of h with

h constant around ∂I

$$f^{-1}(q) \simeq f^{-1}(q') \simeq g^{-1}(q') \simeq g^{-1}(q)$$

q' in a neighborhood of q with only regular values

$h^{-1}(q')$

independence of $q \in S^k \leftrightsquigarrow$ isotopies in S^k

- $M^n \subseteq \mathbb{R}^{n+k}$ closed
- we want $f: S^{n+k} \to S^k$ with $M \cong f^{-1}(q)$

- take a tubular neighborhood $U \subseteq \mathbb{R}^{n+k}$ of M and $\varphi: U \xrightarrow{\sim} M \times \mathbb{R}^k$ with $\varphi(p) = (p, 0)$ and $d\varphi_p X^i_p = e_{i+k} \in \mathbb{R}^{n+k} \cong T_{(p,0)}\mathbb{R}^{n+k}$ for $p \in M$

$$\begin{tikzcd} U \subseteq \mathbb{R}^{m+k} \arrow[r, "\varphi"] \arrow[dr, "f"'] & M \times \mathbb{R}^k \arrow[d, "\pi"] \\ & \mathbb{R}^k \end{tikzcd}$$

- we can find $V \subseteq U$ with $M \subseteq V$ and $f: \mathbb{R}^{n+k} \to \mathbb{R}^k$ with $f|_V = \pi \circ \varphi$, $f(x) \xrightarrow[x \to \infty]{} \infty$ and $f^{-1}(0) = M$

$$f: S^{n+k} \longrightarrow S^k$$